ELASTICITY

A body is said to be rigid if the relative positions of its constituent particles remains unchanged when external deforming forces are applied to it. The nearest approach to a rigid body is diamond or carborundum.

Actually no body is perfectly rigid and every body can be deformed more or less by the application of suitable forces. All these deformed bodies however regain their original shape or size, when the deforming forces are removed.

The property of matter by virtue of which a body tends to regain its original shape and size after the removal of deforming forces is called elasticity.

Some terms related to elasticity :

* **Deforming Force**

   External force which try to change in the length, volume or shape of the body is called deforming force.

* **Perfectly Elastic Body**

   The body which perfectly regains its original form on removing the external deforming force, is defined as a perfectly elastic body. Ex. : quartz – Very nearly a perfect elastic body.

* **Plastic Body**

   (a) The body which does not have the property of opposing the deforming force, is known as a plastic body.

   (b) The bodies which remain in deformed state even after removed of the deforming force are defined as plastic bodies.

* **Internal restoring force**

   When a external force acts at any substance then due to the intermolecular force there is a internal resistance produced into the substance called internal restoring force.

   At equilibrium the numerical value of internal restoring force is equal to the external force.

STRESS

The internal restoring force acting per unit area of cross–section of the deformed body is called stress.

$$\text{Stress} = \frac{\text{Internal restoring force}}{\text{Area of cross section}} = \frac{F_{internal}}{A} = \frac{F_{external}}{A}$$

Stress depends on direction of force as well as direction of area of application so it is tensor.

SI Unit : $N-m^{-2}$

**Dimensions :** $M^1 \ L^{-1} \ T^{-2}$

There are three types of stress :-

(a) Longitudinal Stress : When the stress is normal to the surface of body, then it is known as longitudinal stress. There are two types of longitudinal stress

(i) Tensile Stress : The longitudinal stress, produced due to increase in length of a body, is defined as tensile stress.
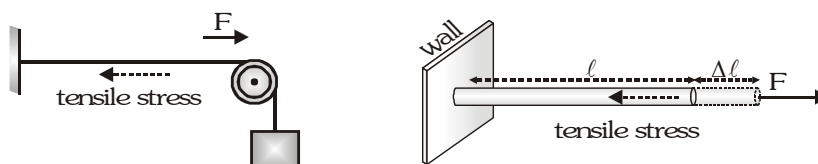
(ii) **Compressive stress :** The longitudinal stress, produced due to decrease in length of a body, is defined as compressive stress.
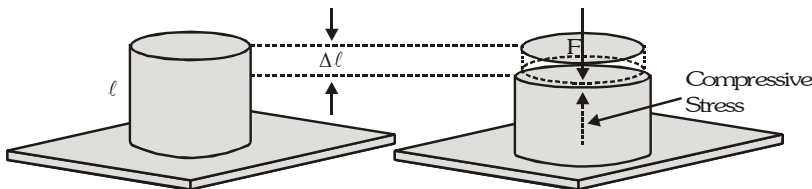

(b) **Volume Stress**

If equal normal forces are applied every one surface of a body, then it undergoes change in volume. The force opposing this change in volume per unit area is defined as volume stress.

(c) **Tangential Stress or Shear Stress**

When the stress is tangential or parallel to the surface of a body then it is known as shear stress. Due to this stress, the shape of the body changes or it gets twisted.
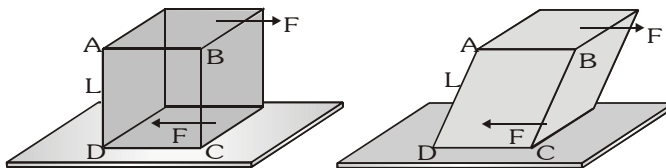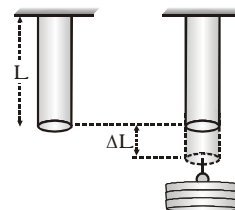

STRAIN

The ratio of change of any dimension to its original dimension is called strain.

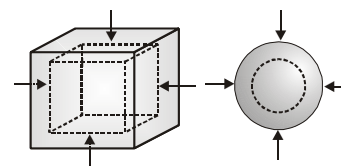$$\text{Strain} = \frac{\text{change in size of the body}}{\text{original size of the body}}$$

It is a unitless and dimensionless quantity.

There are three types of strain :   Type of strain depends upon the directions of applied force.

(a) **Longitudinal strain** $= \dfrac{\text{change in length of the body}}{\text{initial length of the body}} = \dfrac{\Delta L}{L}$

(b) **Volume strain** $= \dfrac{\text{change in volume of the body}}{\text{original volume of the body}} = \dfrac{\Delta V}{V}$

(c) **Shear strain**

When a deforming force is applied to a body parallel to its surface   then its shape (not size) changes. The strain produced in this way is known as shear strain. The strain produced due to change of shape of the body

is known as shear strain. $\tan \phi = \dfrac{\ell}{L}$ or $\phi = \dfrac{\ell}{L} = \dfrac{\text{displacement of upper face}}{\text{distance between two faces}}$

E
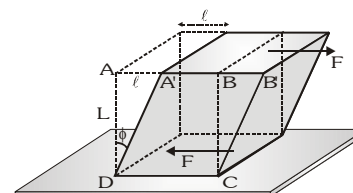
## Relation Between angle of twist and Angle of shear

When a cylinder of length 'ℓ' and radius 'r' is fixed at one end and tangential force is applied at the other end, then the cylinder gets twisted. Figure shows the angle of shear ABA' and angle of twist AOA'. Arc AA' = r θ and Arc AA' = $\ell\phi$ so rθ = $\ell\phi$ $\Rightarrow$ $\phi = \dfrac{r\theta}{\ell}$ where θ = angle of twist, ϕ = angle of shear

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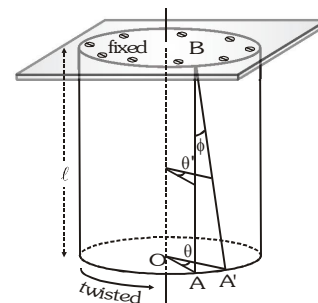
GOLDEN KEY POINTS

- When a material is under tensile stress restoring force are caused by intermolecular attraction while under compressive stress, the restoring force are due to intermolecular repulsion.

- If the deforming force is inclined to the surface at an angle θ such that $\theta \neq 0$ and $\theta \neq 90$ then both tangential and normal stress are developed.

- Linear strain in the direction of force is called **longitudinal** strain while in a direction perpendicular to force **lateral strain**.

Stress – Strain Graph
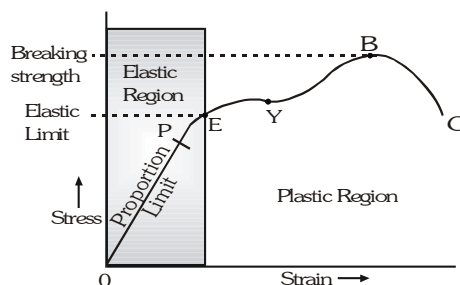
Proportion Limit :

The limit in which Hook's law is valid and stress is directly proportional to strain is called proportion limit. Stress ∝ Strain

Elastic limit :

That maximum stress which on removing the deforming force makes the body to recover completely its original state.

Yield Point :

The point beyond elastic limit, at which the length of wire starts increasing without increasing stress, is defined as the yield point.

Breaking Point :

The position when the strain becomes so large that the wire breaks down at last, is called breaking point. At this position the stress acting in that wire is called breaking stress and strain is called breaking strain.

Elastic after effect

We know that some material bodies take some time to regain their original configuration when the deforming force is removed. The delay in regaining the original configuration by the bodies on the removal of deforming force is called elastic after effect. The elastic after effect is negligibly small for quartz fibre and phosphor bronze. For this reason, the suspensions made from quartz and phosphor-bronze are used in galvanometers and electrometers. For glass fibre elastic after effect is very large. It takes hours for glass fibre to return to its original state on removal of deforming force.

Elastic Fatigue :

The loss of strength of the material due to repeated strains on the material is called elastic fatigue. That is why bridges are declared unsafe after a long time of their use.

Creep :

If a small force is applied for a long time then it causes breaking of metal. For example A fan is hung for 200 years then the shaft will break.

Elastic Hysteresis :

The strain persists even when the stress is removed. This lagging behind of strain is called elastic hysteresis. This is the reason why the values of strain for same stress are different while increasing the load and while decreasing the load.
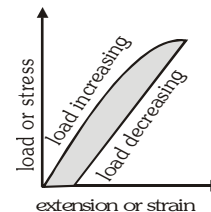
Breaking Stress :

The stress required to cause actual facture of a material is called the breaking stress Breaking stress = F/A

GOLDEN KEY POINTS

- Breaking stress also measures the tensile strength.
- Metals with small plastic deformation are called brittle.
- Metals with large plastic deformation are called ductile.
- Elasticity restoring forces are strictly conservative only when the elastic hysteresis is zero. i.e. the loading and unloading stress – strain curves are identical.
- The material which have low elastic hysteresis have also low elastic relaxation time.
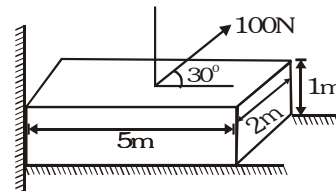
Example

Find out longitudinal stress and tangential stress on a fixed block.

Solution

Longitudinal or normal stress $\sigma_1 = \dfrac{100\sin 30^0}{5 \times 2} = 5 \ \text{N/m}^2$

Tangential stress $\sigma_2 = \dfrac{100\cos 30^0}{5 \times 2} = 5\sqrt{3}\text{N}/\text{m}^2$

Example

The breaking stress of aluminium is 7.5 $\times$ 10^8 dyne cm^{-2}. Find the greatest length of aluminium wire that can hang vertically without breaking. Density of aluminium is 2.7 g cm^{-3}. [Given : g = 980 cm s^{-2}]

Solution

Let ℓ be the greatest length of the wire that can hang vertically without breaking.

Mass of wire m = cross–sectional area (A) $\times$ length (ℓ) $\times$ density (ρ), Weight of wire = mg = A$\ell\rho$g

This is equal to the maximum force that the wire can withstand.

$\therefore$ Breaking stress $= \dfrac{\ell A\rho g}{A} = \ell\rho g \Rightarrow 7.5 \times 10^8 = \ell \times 2.7 \times 980$

$\Rightarrow \ell = \dfrac{7.5 \times 10^8}{2.7 \times 980}$ cm $= 2.834 \times 10^5$ cm$= 2.834$ km

Hooke's Law

If the deformation is small, the stress in a body is proportional to the corresponding strain, this fact is known as Hooke's Law. Within elastic limit : stress $\propto$ strain $\Rightarrow$ $\dfrac{\text{stress}}{\text{strain}}$ =constant

This constant is known as modulus of elasticity or coefficient of elasticity.

The modulus of elasticity depends only on the type of material used. It does not depend upon the value of stress and strain.
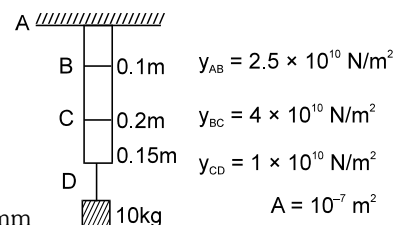
Example

Find out the shift in point B, C and D

Sol. $\Delta L_B = \Delta L_{AB} = \dfrac{FL}{AY} = \dfrac{MgL}{AY} = \dfrac{10 \times 10 \times 0.1}{10^{-7} \times 2.5 \times 10^{10}}$ =4 10^{-3} m = 4mm

$\Delta L_C = \Delta L_B + \Delta L_{BC}$ =4 10^{-3} + $\dfrac{100 \times 0.2}{10^{-7} \times 4 \times 10^{10}}$ =4 10^{-3} + 5 10^{-3} =9mm

$\Delta L_D = \Delta L_C + \Delta L_{CD}$ = 9 10^{-3} + $\dfrac{100 \times 0.15}{10^{-7} \times 1 \times 10^{10}}$ =9 10^{-3} + 15 10^{-3} =24 mm

Diagram labels:
A (fixed)
B 0.1m $y_{AB} = 2.5 \times 10^{10}$ N/m²
C 0.2m $y_{BC} = 4 \times 10^{10}$ N/m²
0.15m $y_{CD} = 1 \times 10^{10}$ N/m²
D
10kg $A = 10^{-7}$ m²

Young's Modulus of Elasticity 'Y'

Within elastic limit the ratio of longitudinal stress and longitudinal strain is called Young's modulus of elasticity.

$$Y = \dfrac{\text{longitudinal stress}}{\text{longitudinal strain}} = \dfrac{F/A}{\ell/L} = \dfrac{FL}{\ell A}$$

Within elastic limit the force acting upon a unit area of a wire by which the length of a wire becomes double, is equivalent to the Young's modulus of elasticity of material of a wire. If L is the length of wire, r is radius and ℓ is the increase in length of the wire by suspending a weight Mg at its one end then Young's modulus

of elasticity of the material of wire $Y = \dfrac{\left(Mg/\pi r^2\right)}{\left(\ell/L\right)} = \dfrac{MgL}{\pi r^2 \ell}$

Unit of Y : N/m² Dimensions of Y : $M^1 L^{-1} T^{-2}$

Increment of length due to own weight

Let a rope of mass M and length L is hanged vertically. As the tension of different point on the rope is different. Stress as well as strain will be different at different point.

(i) maximum stress at hanging point (ii) minimum stress at lower point

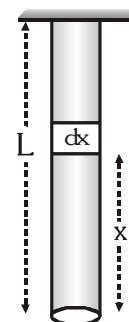Consider a dx element of rope at x distance from lower end then tension $T = \left(\dfrac{M}{L}\right) x\, g$

So stress = $\dfrac{T}{A}$ = $\left(\dfrac{M}{L}\right)\dfrac{xg}{A}$

Let increase in length of dx is dy then strain = $\dfrac{dy}{dx}$

So Young modulus of elasticity $Y = \dfrac{\text{stress}}{\text{strain}} = \dfrac{\dfrac{M}{L}\dfrac{xg}{A}}{dy/dx} \Rightarrow \left(\dfrac{M}{L}\right)\dfrac{xg}{A}dx = Y\,dy$

For full length of rope $\dfrac{Mg}{LA} \int_0^L x\,dx$ =$Y\int_0^{\Delta\ell} dy$ $\Rightarrow \dfrac{Mg}{LA}\dfrac{L^2}{2} = Y\Delta\ell \Rightarrow \Delta\ell = \dfrac{MgL}{2AY}$

[Since the stress is varying linearly we may apply average method to evaluate strain.]

Example

A thin uniform metallic rod of length 0.5 m and radius 0.1 m rotates with an angular velocity 400 rad/s in a horizontal plane about a vertical axis passing through one of its ends. Calculate (a) tension in the rod and (b) the elongation of the rod. The density of material of the rod is 10^4 kg/m^3 and the Young's modulus is 2 10^{11} N/m^2.

Solution

(a) Consider an element of length dr at a distance r from the axis of rotation as shown in figure. The centripetal force acting on this element will be $dT = dmr\omega^2 = (\rho A dr) r\omega^2$.

As this force is provided by tension in the rod (due to elasticity), so the tension in the rod at a distance r from the axis of rotation will be due to the centripetal force due to all elements between x = r to x = L   i.e.,
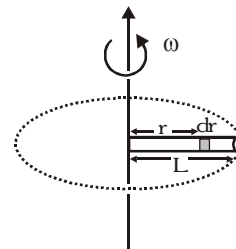
$$T = \int_r^L \rho A\omega^2 r\,dr = \frac{1}{2}\rho A\omega^2 \left[L^2 - r^2\right] \dots(i)$$

So here $\quad T = \frac{1}{2} \quad 10^4 \quad \pi \quad 10^{-2} \quad (400)^2 \left[\left(\frac{1}{2}\right)^2 - r^2\right] = 8\pi \times 10^6 \left[\frac{1}{4} - r^2\right] N$

(b) Now if dy is the elongation in the element of length dr at position r then strain

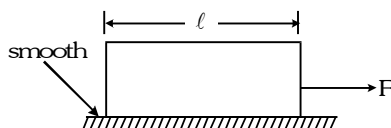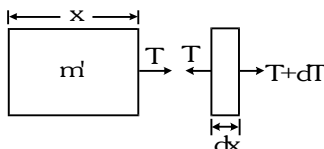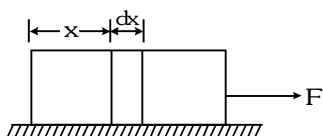$$\frac{dy}{dr} = \frac{stress}{Y} = \frac{T}{AY} = \frac{1}{2}\frac{\rho\omega^2}{Y}\left[L^2 - r^2\right]$$

So the elongation of the whole rod

$$\Delta L = \frac{\rho\omega^2}{2Y}\int_0^L \left(L^2 - r^2\right)dr = \frac{1}{3}\frac{\rho\omega^2 L^3}{Y} = \frac{1}{3}\frac{10^4 \times (400)^2 (0.5)^3}{2 \times 10^{11}} = \frac{1}{3}\times 10^{-3}\,m$$

Example

Find out the elongation in block. If mass, area of cross-section and young modulus of block are m, A and y respectively.

Solution

Acceleration, $\quad a = \frac{F}{m}\quad$ then $T = m'a\quad$ where $\Rightarrow m' = \frac{m}{\ell}x;\quad T = \frac{m}{\ell}x\,\frac{F}{m} = \frac{Fx}{\ell}$

Elongation in element 'dx' $= \frac{Tdx}{Ay}$,$\qquad$ total elongation, $\delta = \int_o^\ell \frac{Tdx}{Ay} = \int_o^\ell \frac{Fxdx}{A\ell y} = \frac{F\ell}{2Ay}$

E

Bulk modulus of elasticity 'K' or 'B'

Within elastic limit the ratio of the volume stress and the volume strain is called bulk modulus of elasticity.

$$K \text{ or } B = \frac{volume\ stress}{volume\ strain} = \frac{F/A}{\dfrac{-\Delta V}{V}} = \frac{\Delta P}{\dfrac{-\Delta V}{V}}$$

The minus sign indicates a decrease in volume with an increase in stress.

Unit of K : N m^{-2} or pascal

Bulk modulus of an ideal gas is process dependence.

For isothermal process PV = constant $\Rightarrow$ PdV + VdP = 0 $\Rightarrow$ $P = \dfrac{-dP}{dV/V}$ So bulk modulus = P

For adiabatic process PV^{γ} = constant $\Rightarrow \gamma PV^{\gamma-1}dV + V^{\gamma}dP = 0 \Rightarrow \gamma PdV + VdP = 0 \Rightarrow \gamma P = \dfrac{-dP}{dV/V}$

So bulk modulus = γP

For any polytropic process PV^{n} = constant $\Rightarrow nPV^{n-1}dV | V^{n}dP = 0 \Rightarrow PdV + VdP = 0 \Rightarrow nP = \dfrac{-dP}{dV/V}$

So bulk modulus = nP

Compressibility : The reciprocal of bulk modulus of elasticity is defined as compressibility. $C = \dfrac{1}{K}$
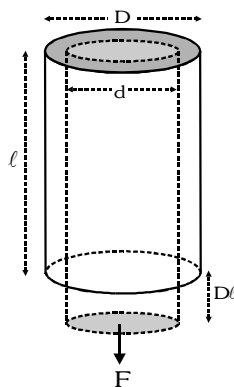
Modulus of Rigidity 'η'

Within elastic limit the ratio of shearing stress and shearing strain is called modulus of rigidity of a material.

$$\eta = \frac{shearing\ stress}{shearing\ strain} = \left(\frac{\dfrac{F_{tangential}}{A}}{\phi} \right) = \frac{F_{tangential}}{A\phi}$$

Note : Angle of shear 'ϕ' always take in radian

Poisson's Ratio (σ)

In elastic limit, the ratio of lateral strain and longitudinal strain is called Poisson's ratio.

$$\sigma = \frac{lateral\ strain}{longitudinal\ strain} = \frac{\beta}{\alpha}, \text{ where } \beta = \frac{-\Delta D}{D} = \frac{d-D}{D} \text{ \& } \alpha = \frac{\Delta L}{L}$$

Work done in stretching a wire (Potential energy of a stretched wire)

When a wire is stretched, work is done against the interatomic forces, which is stored in the form of elastic potential energy.

For a wire of length L_o stretched by a distance x, the restoring elastic force is :

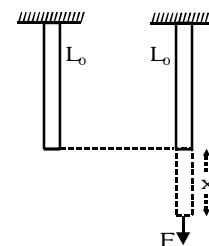$$F = \text{stress} \times \text{area} = Y\left[\frac{x}{L_o}\right]A$$

The work has to be done against the elastic restoring forces in stretching dx

$$dW = F\,dx = \frac{YA}{L_0}x\,dx$$

The total work done in stretching the wire from $x = 0$ to $x = \Delta\ell$ is, then

$$W = \int_0^{\Delta\ell}\frac{YA}{L_o}x\,dx = \frac{YA}{L_o}\left[\frac{x^2}{2}\right]_0^{\Delta\ell} = \frac{YA(\Delta\ell)^2}{2L_o}$$

$$W = \frac{1}{2} \times Y \times (\text{strain})^2 \times \text{original volume} = \frac{1}{2}(\text{stress})\,(\text{strain})\,(\text{volume})$$

GOLDEN KEY POINTS
•   The value of K is maximum for solids and minimum for gases.
•   For any ideal rigid body all three elastic modulus are infinite .
• η is the characteristic of solid material only as the fluids do not have fixed shape.
•   Potential energy density = area under the stress–strain curve.
•   Young's modulus = Slope of the stress–strain curve
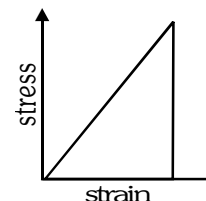
Example

A steel wire of 4.0 m in length is stretched through 2.00 mm. The cross–sectional area of the wire is 2.0 mm^2. If Young's modulus of steel is 2.0 $\times$ 10^{11} N/m^2 find (i) the energy density of wire (ii) the elastic potential energy stored in the wire.

Solution

(i) The energy density of stretched wire

$$=\frac{1}{2}\times\text{stress}\times\text{strain}=\frac{1}{2}\times Y\times(\text{strain})^2=\frac{1}{2}\times 2.0\times 10^{11}\left[\frac{2\times10^{-3}}{4}\right]^2 = 2.5\times10^4\ \text{J/m}^3$$

(ii) Elastic potential energy = energy density $\times$ volume = 2.5 $\times$ 10^4 $\times$ (2.0 $\times$ 10^{-6}) $\times$ 4.0 J = 20 $\times$ 10^{-2} = 0.20J

Example

Find the depth of lake at which density of water is 1% greater than at the surface. Given compressibility $K = 50 \times 10^{-6}$ /atm.
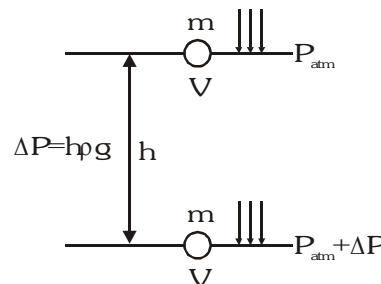
Solution

$$B = \frac{\Delta P}{-\Delta V / V} \Rightarrow \frac{\Delta V}{V} = -\frac{\Delta P}{B}$$

We know $P = P_{atm} + h\rho g$ and $m = \rho V = $ constant

$$d\rho V + dV \rho = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V} \text{ i.e. } \frac{\Delta \rho}{\rho} = \frac{\Delta P}{B} \Rightarrow \frac{\Delta \rho}{\rho} = \frac{1}{100} = \frac{h\rho g}{B}$$

[assuming ρ = constant]; $h\rho g = \frac{B}{100} = \frac{1}{100K} \Rightarrow h\rho g = \frac{1 \times 1 \times 10^5}{100 \times 50 \times 10^{-6}}$

$$h = \frac{10^5}{5000 \times 10^{-6} \times 1000 \times 10} = \frac{100 \times 10^3}{50} m = 2km$$
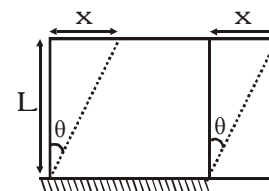
Example

A rubber cube of side 5 cm has one side fixed while a tangential force equal to 1800 N is applied to opposite face. Find the shearing strain and the lateral displacement of the strained face. Modulus of rigidity for rubber is 2.4×10^6 N/m^2.

Solution

$$\text{Stress} = \frac{F}{A} = \eta \frac{x}{L}$$

$$\text{Strain} = \theta = \frac{F}{A\eta} = \frac{1800}{25 \times 10^{-4} \times 2.4 \times 10^6} = \frac{180}{25 \times 24} = \frac{3}{10} = 0.3 \text{ rad}$$

as $\frac{x}{L} = 0.3 \Rightarrow x = 0.3 \times 5 \times 10^{-2} = 1.5 \times 10^{-2}$ m = 1.5 mm

Analogy of Rod as a spring

$$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow Y = \frac{F\ell}{A\Delta\ell} \text{ or } F = \frac{AY}{\ell}\Delta\ell$$

$\frac{AY}{\ell}$ = constant, depends on type of material and geometry of rod.

$F = k\Delta\ell$ where $k = \frac{AY}{\ell}$ = equivalent spring constant.

For the system of rods shown in figure (a), the replaced spring system is shown in figure (b) [two spring in series]. Figure (c) represents equivalent spring system.
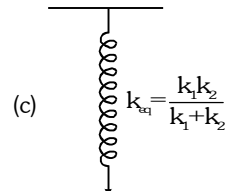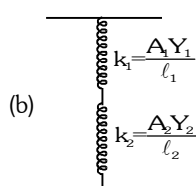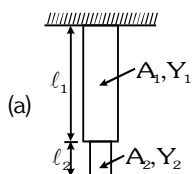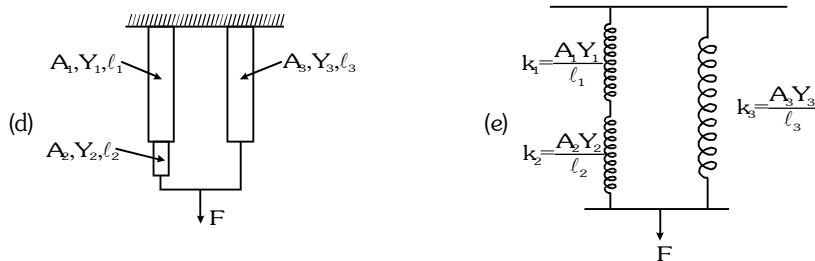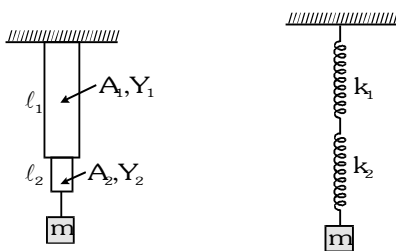
Figure (d) represents another combination of rods and their replaced spring system.

(d)

(e)

Example

A mass 'm' is attached with rods as shown in figure. This mass is slightly stretched and released whether the motion of mass is S.H.M., if yes then find out the time period.
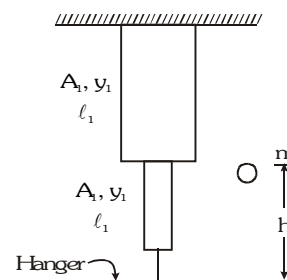

Solution

Here $k_{eq} = \dfrac{k_1 k_2}{k_1 + k_2}$ and $T = 2\pi \sqrt{\dfrac{m}{k_{eq}}} = 2\pi \sqrt{\dfrac{m(k_1 + k_2)}{k_1 k_2}}$ where $k_1 = \dfrac{A_1 Y_1}{\ell_1}$ and $k_2 = \dfrac{A_2 Y_2}{\ell_2}$

Example

Hanger is mass-less. A ball of mass 'm' drops from a height 'h', which sticks to hanger after striking. Neglect over turning, find out the maximum extension in rod. Assuming rod is massless. Let maximum extension be x_{max}.

Sol. Applying energy conservation $mg (h + x_{max}) = \dfrac{1}{2} \dfrac{k_1 k_2}{k_1 + k_2} x^2$

where $k_1 = \dfrac{A_1 y_1}{\ell_1}$; $k_2 = \dfrac{A_2 y_2}{\ell_2}$ & $K_{eq} = \dfrac{A_1 A_2 y_1 y_2}{A_1 y_1 \ell_2 + A_2 y_1 \ell_1}$

therefore $k_{eq} x_{max}^2 - 2mg x_{max} - 2mgh = 0$

$x_{max} = \dfrac{2mg \pm \sqrt{4m^2 g^2 + 8mgh k_{eq}}}{2k_{eq}} = \dfrac{mg}{k_{eq}} + \sqrt{\dfrac{m^2 g^2}{k_{eq}^2} + \dfrac{2mgh}{k_{eq}}}$
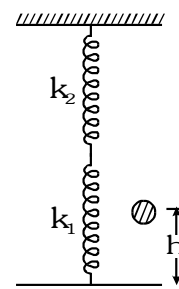
Otherway by S.H.M.

$$\omega = \sqrt{\frac{k_{eq}}{m}} \text{ and } v = \omega \sqrt{a^2 - y^2}$$

$$\Rightarrow \sqrt{2gh} = \sqrt{\frac{keq}{m}}\sqrt{a^2 - y^2} \Rightarrow \sqrt{\frac{2mgh}{k_{eq}} + \frac{m^2g^2}{k_{eq}^2}} = a$$

$$\text{Maximum extension} = a + y = \frac{mg}{k_{eq}} + \sqrt{\frac{m^2g^2}{k_{eq}} + \frac{2mgh}{k_{eq}}}$$

Thermal Stress :

If temperature of rod is increased by ΔT, then change in length $\Delta \ell = \ell \ \alpha \ \Delta T$; strain $= \dfrac{\Delta \ell}{\ell} = \alpha \Delta T$
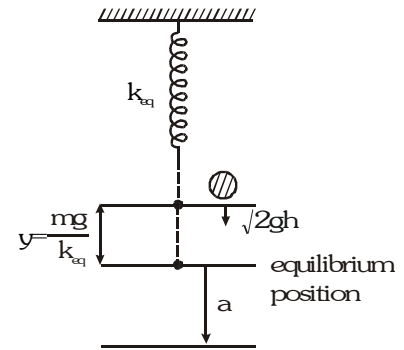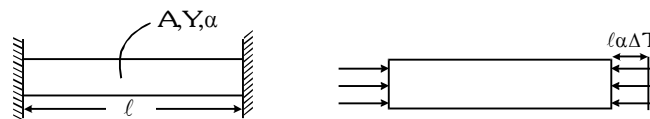
But due to rigid support, there is no strain. Supports provide force on stresses to keep the length of rod same

$$Y = \frac{stress}{strain}$$
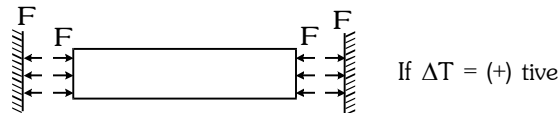
If $\Delta T = (+)$ tive

Thermal stress $= y$ strain $= Y\alpha \ \Delta T$
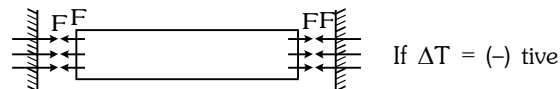
If $\Delta T = (-)$ tive

$$\frac{F}{A} = Y \alpha \Delta T \qquad\qquad F = AY\alpha \ \Delta T$$

Example

When composite rod is free, then composite length increases to 2.002 m for temperature 20 C to 120 C. When composite rod is fixed between the support, there is no change in component length find y and α of steel, if $y_c = 1.5 \times 10^{13}$ N/m^2 $\alpha_c = 1.6 \times 10^{-5}$/ C.
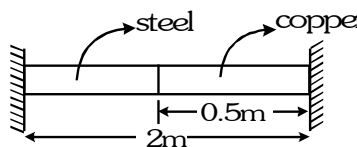
Solution

$$\Delta \ell = \ell_s \ \alpha_s \ \Delta T + \ell_c \ \alpha_c \Delta T \Rightarrow 0.002 = [1.5 \ \alpha_s + 0.5 \times 1.6 \times 10^{-5}] \times 100 \qquad \Rightarrow \alpha_s = \frac{1.2 \times 10^{-5}}{1.5} = 8 \times 10^{-6}/ C$$

there is no change in component length
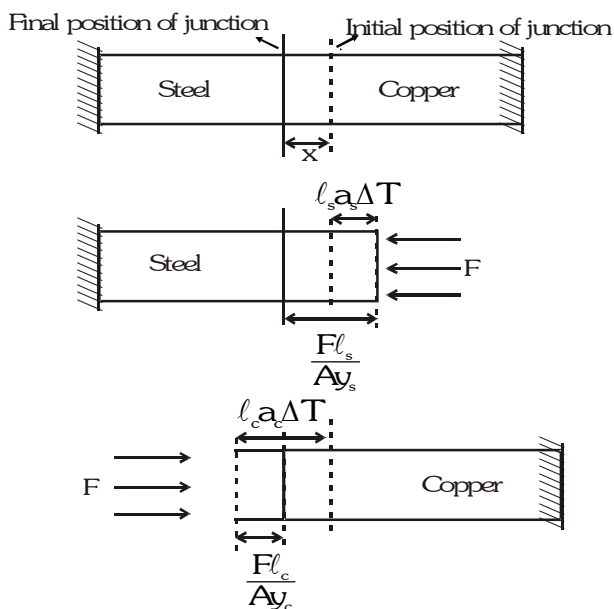
For steel $\qquad x = \ell_s\,\alpha_s\,\Delta T - \dfrac{F\ell_s}{AY_s} = 0 \Rightarrow \dfrac{F}{AY_s} = \alpha_s\,\Delta T$ $\qquad$ ....(i)

For copper $\qquad x = \dfrac{F\ell_c}{AY_c} - \ell_c\,\alpha_c\,\Delta T = 0 \Rightarrow \dfrac{F}{Ay_c} = \alpha_c\,\Delta T$ $\qquad$ ...(ii)

(ii)/(i) $\Rightarrow Y_s = Y_c\,\dfrac{\alpha_c}{\alpha_s} = \dfrac{1.5 \times 10^{13} \times 16 \times 10^{-5}}{8 \times 10^{-6}}$

$Y_s = 3 \quad 10^{13}$ N/m^2

Applications of Elasticity

Some of the important applications of the elasticity of the materials are discussed as follows :

The material used in bridges lose its elastic strength with time bridges are declared unsafe after long use.

To estimate the maximum height of a mountain :

The pressure at the base of the mountain = $h\rho g$ = stress.

The elastic limit of a typical rock is $3 \quad 10^8$ N m^{-2}

The stress must be less than the elastic limits, otherwise the rock begins to flow.

$h < \dfrac{3 \times 10^8}{\rho g} < \dfrac{3 \times 10^8}{3 \times 10^3 \times 10} < 10^4$ m ($\because \rho = 3 \quad 10^3$ kg m^{-3} ; g = 10 ms^{-2}) $\quad$ or $\quad$ h = 10 km

It may be noted that the height of Mount Everest is nearly 9 km.

Torsion constant of a wire

$C = \dfrac{\pi \eta r^4}{2\ell}$ Where η is modulus of rigidity r and ℓ is radius and length of wire respectively.

(a) Toque required for twisting by angle θ, $\tau = C\theta$.

(b) Work done in twisting by angle θ, $W = \dfrac{1}{2} C\theta^2$.

Effect of Temperature on elasticity

When temperature is increased then due to weakness of inter molecular force the elastic properties in general decreases i.e. elastic constant decreases. Plasticity increases with temperature. For example, at ordinary room temperature, carbon is elastic but at high temperature, carbon becomes plastic. Lead is not much elastic at room temperature but when cooled in liquid nitrogen exhibit highly elastic behaviour.

For a special kind of steel, elastic constants do not vary appreciably temperature. This steel is called 'INVAR steel'.

Effect of Impurity on elasticity

Y is slightly increase by impurity. The inter molecular attraction force inside wire effectively increase by impurity due to this external force can be easily opposed.

SURFACE TENSION

Surface tension is basically a property of liquid. The liquid surface behaves like a stretched elastic membrane which has a natural tendency to contract and tends to have a minimum possible area. This property of liquid is called *surface tension*.

INTERMOLECULAR FORCES

l   The force which acts between the atoms or the molecules of different substances is called intermolecular force. This force is of two types.

   (a) Cohesive force – The force acting between the molecules of one type of molecules of same substance is called cohesive force.

   (b) Adhesive force – The force acting between different types of molecules or molecules of different substance is called adhesive force.

l   Intermolecular forces are different from the gravitational forces not obey the inverse–square law

l   The distance upto which these forces effective, is called molecular range. This distance is nearly 10^{-9} m. Within this limit this increases very rapidly as the distance decreases.

l   Molecular range depends on the nature of the substance

Examples:

•   **Water Glass :** Water wets glass surface but mercury does not. Because when water comes in contact with glass the adhesive force acts between water and glass molecules. As adhesive force is greater than the cohesive force of water molecules, the water molecules, cling with glass surface and surface become wet. In case of mercury adhesive force is less than that of cohesive force and Hg–molecules do not cling with glass surface and surface does not wet with Hg–molecules.

•   **Oil–water :** Since Cohesive force of water >Adhesive force oil–water> Cohesive force of oil.

   (i)   If water drop is poured on the surface of oil, it contracts in the form of globule.

   (ii) If oil drop is poured on the surface of water it spreads to a larger area in the form of thin film.

•   **Ink–paper :** Since adhesive force between ink–paper > cohesive force on ink, the ink sticks to the paper.

EXPLANATION OF SURFACE TENSION (Molecular theory of surface Tension)

   Laplace firstly explained the phenomenon of surface tension on the basis of intermolecular forces. According to him surface tension is a molecular phenomenon and its root causes are electromagnetic forces. He explained the cause of surface tension as described under. If the distance between two molecules is less than the molecular range C ($\approx 10^{-9}$ m) then they attract each other, but if the distance is more than this then the attraction becomes negligible. If a sphere of radius C with a molecule at centre is drawn, then only those molecules which are enclosed within this sphere can attract or be attracted by the molecule at the centre of the sphere. This sphere is called sphere of molecular activity or sphere of influence. In order to understand the tension acting at the free surface of liquid, let us consider four liquid molecules like A, B, C and D along with their spheres of molecular activity.

(a)   According to figure D sphere is completely inside liquid. So molecule is attracted equally in all directions and hence resultant force is equal to zero.

(b)   According to figure sphere of molecule C is just below the liquid surface. So resultant force is equal to zero.

(c)   The molecule B which is a little below the liquid surface is more attracted downwards due to excess of molecules downwards. Hence the resultant force is acting downwards.

E

(d)   Molecule A is situated at surface so that its sphere of molecular activity is half outside the liquid and half inside. Only down portion has liquid molecules. Hence it experiences a maximum downward force. Thus all the molecules situated between the surface and a plane XY, distant C below the surface, experience a resultant downward cohesive force. When the surface area of liquid is increased molecules from the interior of the liquid rise to the surface.
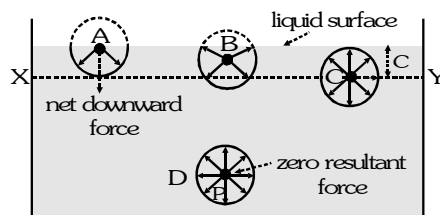
As these molecules reach near the surface, work is done against the downward cohesive force. This work is stored in the molecules in the form of potential energy. Thus the potential energy of the molecules lying in the surface is greater than that of the molecules in the interior of the liquid. A system is in stable equilibrium when its potential energy is minimum. Hence in order to have minimum potential energy the liquid surface tends to have minimum number of molecules in it. In other words the surfaces tends to contract to a minimum possible area. This tendency is exhibited as surface tension.

GOLDEN KEY POINTS

*   Surface tension is a scalar quantity.

*   It acts tangential to liquid surface.

*   Surface tension is always produced due to cohesive force.

*   More is the cohesive force, more is the surface tension.

*   When surface area of liquid is increased, molecules from the interior of the liquid rise to the surface. For this, work is done against the downward cohesive force.

     As a result, its potential energy increases and internal energy decreases. So on increase in surface area cooling occurs. If liquid temperature remains same, then extra energy may be given by external agency. So the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called Surface Energy, free surface energy or surface energy density.

DEPENDENCY OF SURFACE TENSION

1   On Cohesive Force

Those factors which increase the cohesive force between molecules increase the surface tension and those which decrease the cohesive force between molecules decrease the surface tension.

1   On Impurities

If the impurity is completely soluble then on mixing it in the liquid, its surface tension increases. e.g., on dissolving ionic salts in small quantities in a liquid, its surface tension increases. If the impurity is partially soluble in a liquid then its surface tension decreases because adhesive force between insoluble impurity molecules and liquid molecules decreases cohesive force effectively, e.g.

(a) On mixing detergent in water its surface tension decreases.

(b) Surface tension of water is more than (alcohol + water) mixture.

1   On Temperature

On increasing temperature surface tension decreases. At critical temperature and boiling point it becomes zero.

Note : Surface tension of water is maximum at 4 C
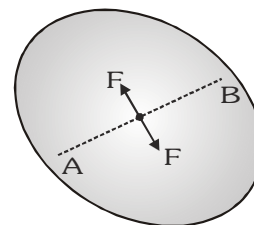
1   On Contamination

The dust particles or lubricating materials on the liquid surface decreases its surface tension.

1   On Electrification

The surface tension of a liquid decreases due to electrification because a force starts acting due to it in the outward direction normal to the free surface of liquid.

DEFINITION OF SURFACE TENSION

Surface tension can be defined in the form of an imaginary line on the surface or by relating it to the work done. The force acting per unit length of an imaginary line drawn on the free liquid surface at right angles to the line and in the plane of liquid surface, is defined as surface tension. Let an imaginary line AB be drawn in any direction on a liquid surface. The surface on either side of this line exerts a pulling force, which is perpendicular to line AB. If force is F and length of AB is L then $T = \dfrac{F}{L}$
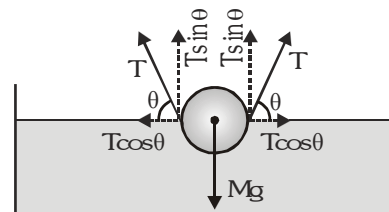
SI UNITS : N/m and J/m^2 **CGS UNITS :** dyne/cm and erg/cm^2 **Dimensions :** $M^1 L^0 T^{-2}$

Illustrations:

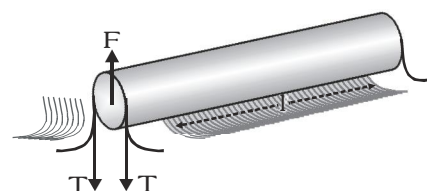- When any needle floats on the liquid surface then $2T\ell \sin\theta = mg$

  Ex. A mosquito sitting on a liquid surface.

- If the needle is lifted from the liquid surface then required excess force will be $F_{excess} = 2T\ell$

  Minimum force required $F_{min} = mg + 2T\ell$

- Required excess force for a circular thick ring (or hollow disc) having internal and external radii r_1 and r_2 is dipped in and taken out from liquid $F_{excess} = F_1 + F_2 = T(2\pi\, r_1) + T(2\pi\, r_2) = 2\pi T(r_1 + r_2)$
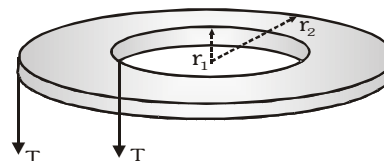
- Required excess force for a circular ring ($r_1 = r_2 = r$)

  $$F_{excess} = 2\pi\, T(\, r + r) = 4\,\pi\, r\, T$$
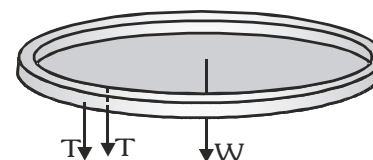
- Required excess force for a circular disc ($r_1 = 0$, $r_2 = r$)

  $$F_{excess} = 2\,\pi\, rT$$

Surface Energy

According to molecular theory of surface tension the molecule in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called ' Surfa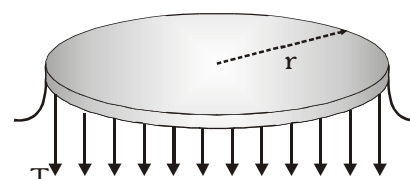ce energy". Let a liquid film be formed on a wire frame and a straight wire of length ℓ can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solution, each surface will pull the wire parallel to itself with a force $T\ell$.
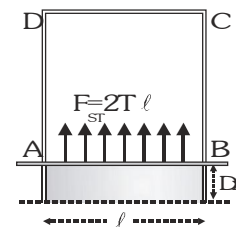
Thus, net force on the wire due to both the surfaces is $2T\ell$. Apply an external force F equal and opposite to it to keep the wire is equilibrium. Thus, $F = 2T\ell$ Now, suppose the wire is moved through a small distance dx, the work done by the force is, $dW = F\, dx = (2T\ell)\, dx$ But $(2\ell)\, (dx)$ is the total increase in area of both the surfaces of the film. Let it be dA. Then, $dW = T\, dA \Rightarrow T = dW/dA$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by unity. Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface. $\qquad T = \dfrac{dU}{dA}$ [as $dW = dU$]

Special Cases

- Work done (surface energy) in formation of a drop of radius r = Work done against surface tension

$$W = \text{Surface tension } T \quad \text{change in area } \Delta A = T \quad 4\pi r^2 = 4\pi r^2 T$$

- Work done (surface energy) in formation of a soap bubble of radius r :

$$W = T \quad \Delta A \text{ or } W = T \quad 2 \quad 4\pi r^2 = 8\pi r^2 T \qquad [\because \text{ soap bubble has two surfaces}]$$

Example

The length of a needle floating on water is 2.5 cm. Calculate the added force required to pull the needle out of water. [$T = 7.2 \quad 10^{-2}$ N/m]

Solution

The force of surface tension $F = T \quad 2\ell$ ($\because$ Two free surfaces are there)

$$\Rightarrow F = 7.2 \quad 10^{-2} \quad 2 \quad 2.5 \quad 10^{-2} = 3.6 \quad 10^{-3} \text{ N}$$

Example

A paper disc of radius R from which a hole of radius r is cut out, is floating in a liquid of surface tension, T. What will be force on the disc due to surface tension?

Solution

$$T = \frac{F}{L} = \frac{F}{2\pi(R + r)} \qquad \therefore\ F = 2\pi\,(R + r)T$$

Example

Calculate the work done against surface tension in blowing a soap bubble from a radius 10 cm to 20 cm if the surface tension of soap solution is $25 \quad 10^{-3}$ N/m. Then compare it with liquid drop of same radius.

Solution

(i) For soap bubble : Extension in area $= 2 \quad 4\pi r_2{}^2 - 2 \quad 4\pi r_1{}^2 = 8\pi\ [(0.2)^2 - (0.1)^2] = 0.24\ \pi\ m^2$

Work done $W_1 = $ surface tension $\quad$ extension in area $= 25 \quad 10^{-3} \quad 0.24\ \pi = 6\pi \quad 10^{-3}$ J

(ii) For Liquid Drop : in case of liquid drop only one free surface, so extension in area will be half of soap bubble

$$\therefore\ W_2 = \frac{W_1}{2} = 3\pi \quad 10^{-3} \text{ J}$$

SPLITTING OF BIGGER DROP INTO SMALLER DROPLETS

If bigger drop is spitted into smaller droplets then in this process volume of liquid always remain conserved. Let bigger drop has radius R. It is splitted into n smaller drops of radius r then by conservation of volume

(i) $\frac{4}{3}\pi R^3 = n\left[\frac{4}{3}\pi r^3\right] \Rightarrow n = \left[\frac{R}{r}\right]^3 \Rightarrow r = \frac{R}{n^{1/3}}$

(ii) Initial surface area = $4\pi R^2$ and final surface area = $n(4\pi r^2)$

Therefore initial surface energy $E_i = 4\pi R^2 T$ and final surface energy $E_f = n(4\pi r^2 T)$

Change in area $\Delta A = n4\pi r^2 - 4\pi R^2 = 4\pi (nr^2 - R^2)$

Therefore the amount of surface energy absorbed i.e. $\Delta E = E_f - E_i = 4\pi T (nr^2 - R^2)$

∴ Magnitude of work done against surface tension i.e. $W = 4\pi (nr^2 - R^2)T$

$W = 4\pi T (nr^2 - R^2) = 4\pi R^2 T (n^{1/3} - 1) = 4\pi R^2 T\left[\frac{R}{r} - 1\right] \Rightarrow W = 4\pi R^3 T \left[\frac{1}{r} - \frac{1}{R}\right]$

In this process temperature of system decreases as energy gets absorbed in increasing surface area.

$W = J m s \Delta\theta = 4\pi R^3 T\left[\frac{1}{r} - \frac{1}{R}\right] \Rightarrow \Delta\theta = \frac{4\pi R^3 T}{\frac{4}{3}\pi R^3 J\rho s}\left[\frac{1}{r} - \frac{1}{R}\right] = \frac{3T}{J\rho s}\left[\frac{1}{r} - \frac{1}{R}\right]$

Where ρ = liquid density, s = liquid's specific heat

Thus in this process area increasing, surface energy increasing, internal energy decreasing, temperature decreasing, and energy is absorbed.

Example

A big drop is formed by coalescing 1000 small droplets of water .What will be the change in surface energy. What will be the ratio between the total surface energy of the droplets and the surface energy of the big drop ?

Solution

By conservation of volume $\frac{4}{3}\pi R^3 = 1000 \quad \frac{4}{3}\pi r^3 \Rightarrow r = \frac{R}{10}$

Surface energy of 1000 droplets = 1000 T $4\pi \left[\frac{R}{10}\right]^2$ = 10 (T $4\pi R^2$)

Surface energy of the big drop = T $4\pi R^2$

Surface energy will decrease in the process of formation of bigger drop, hence energy is released and temperature increases . ∴ $\frac{\text{total surface energy of 1000 droplets}}{\text{surface energy of big drop}} = \frac{10(T \times 4\pi R^2)}{T \times 4\pi R^2} = \frac{10}{1}$

Example

A water drop of radius 1mm is broken into 10^6 identical drops. Surface tension of water is 72 dynes/cm. Find the energy spent in this process.

Solution

As volume of water remains constant, so $\frac{4}{3}\pi R^3 = n\frac{4}{3}\pi r^3 \Rightarrow r = \frac{R}{n^{1/3}}$

Increase in surface area $\Delta A = n (4\pi r^2) - 4\pi R^2 = 4\pi(n^{1/3} - 1) R^2 = 4\pi(100 - 1)10^{-6}$

∴ Energy spent = $T\Delta A = 4\pi$ 99 10^{-6} 72 10^{-3} = 89.5 10^{-6} J

EXCESS PRESSURE INSIDE A CURVED LIQUID SURFACE

The pressure on the concave side of curved liquid surface is greater than that on the convex side. Therefore pressure difference exists across two sides of a curved surface. This pressure difference is called *excess pressure*.

Excess pressure inside the drop

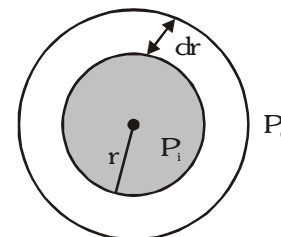Let a drop of radius r having internal and external pressure P_i and P_0 respectively, so that excess pressure

$$P_{ex} = (P_i - P_0).$$

If the radius of the drop is changed from r to (r+dr) then

Work done = F.dr = (PA) dr = P $4\pi r^2 dr$

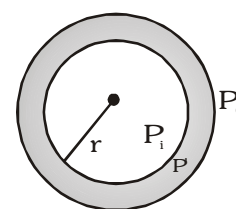Change in surface area = $r\pi(r + dr)^2 - 4\pi r^2 = 8\pi r dr$

So by definition of surface energy $T = \dfrac{W}{\Delta A} = \dfrac{4\pi r^2 P dr}{8\pi r dr} \Rightarrow P_{ex} = (P_i - P_0) = \dfrac{2T}{r}$
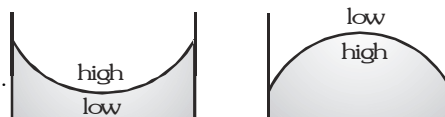
Excess pressure inside soap bubble :

Since the soap bubble has two surfaces. The excess pressure will get doubled as compared to a drop

$P_i - P' = \dfrac{2T}{r}$, $P' - P_0 = \dfrac{2T}{r}$ $\Rightarrow$ excess pressure = $P_i - P_0 = \dfrac{4T}{r}$

GOLDEN KEY POINTS

- For liquid surface, pressure on concave side is always higher than convex side

- If a bubble is formed inside a liquid, the pressure inside the bubble is more than the pressure outside the bubble.

- A soap film is formed in a wire frame. A loop of thread is lying on the film. If the film inside the loop is broken then the tension in the thread will be 2Tr.
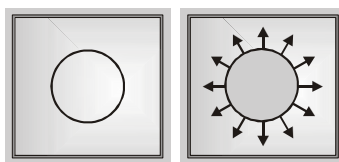
- In the following arrangement, air will flow from bubble A to B if T_2 and T_3 are opened, because pressure in A is greater than in B.
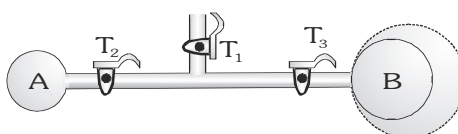
- The force required to separate two plates of Area A is given by $F = \dfrac{2AT}{d}$

Example

Prove that If two bubbles of radii r_1 and r_2 coalesce isothermally in vacuum then the radius of new bubble will be

$$r = \sqrt{r_1^2 + r_2^2}$$

Solution

When two bubbles coalesce then number of molecules of air will remain constant and temperature also constant

so $n_1 + n_2 = n \Rightarrow P_1V_1 + P_2V_2 = PV \Rightarrow \dfrac{4T}{r_1}\left(\dfrac{4}{3}\pi r_1^3\right) + \dfrac{4T}{r_2}\left(\dfrac{4}{3}\pi r_2^3\right) = \dfrac{4T}{r}\left(\dfrac{4}{3}\pi r^3\right) \Rightarrow r = \sqrt{r_1^2 + r_2^2}$
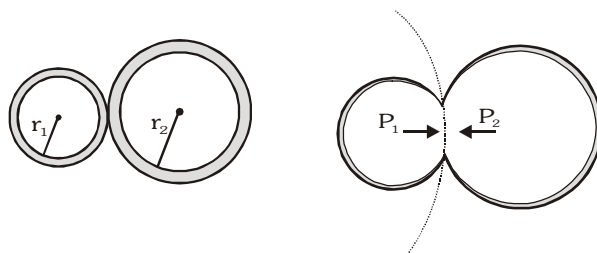
Example

Prove that If two bubbles of radii r_1 and r_2 ($r_1 < r_2$) come in contact with each other then the radius of curvature

of the common surface $r = \dfrac{r_1 r_2}{r_2 - r_1}$.

Solution

$\because r_1 < r_2 \qquad \qquad \therefore P_1 > P_2$ Small part of bubbles is in equilibrium

$\Rightarrow P_1(\Delta A) - P_2(\Delta A) = \dfrac{4T}{r}\Delta A \Rightarrow \dfrac{4T}{r_1} - \dfrac{4T}{r_2} = \dfrac{4T}{r} \Rightarrow r = \dfrac{r_1 r_2}{r_2 - r_1}$

Example

Calculate the excess pressure within a bubble of air of radius 0.1 mm in water. If the bubble had been formed 10 cm below the water surface on a day when the atmospheric pressure was 1.013×10^5 Pa, then what would have been the total pressure inside the bubble? Surface tension of water $= 73 \times 10^{-3}$ N/m

Solution

Excess pressure $P_{excess} = \dfrac{2T}{r} = \dfrac{2 \times 73 \times 10^{-3}}{0.1 \times 10^{-3}} = 1460$ Pa

The pressure at a depth d, in liquid is P = hdg. Therefore, the total pressure inside the air bubble is

$$P_{in} = P_{atm} + hdg + \dfrac{2T}{r} = (1.013 \times 10^5) + (10 \times 10^{-2} \times 10^3 \times 9.8) + 1460$$

$$= 101300 + 980 + 1460 = 103740 = 1.037 \times 10^5 \text{ Pa}$$

Example

The limbs of a manometer consist of uniform capillary tubes of radii are 10^{-3} & 7.2×10^{-4} m. Find out the correct pressure difference if the level of the liquid in narrower tube stands 0.2 m above that in the broader tube.(Density of liquid $=10^3$ kg/m^3, Surface tension $= 72 \times 10^{-3}$ N/m)

E

Solution

If P_1 and P_2 are the pressures in the broader and narrower tubes of radii r_1 and r_2 respectively, the pressure just below the meniscus in the respective tubes will be $P_1 - \dfrac{2T}{r_1}$ and $P_2 - \dfrac{2T}{r_2}$

So that $\left[P_1 - \dfrac{2T}{r_1}\right] - \left[P_2 - \dfrac{2T}{r_2}\right] = h\rho g$ or $P_1 - P_2 = h\rho g - 2T\left[\dfrac{1}{r_2} - \dfrac{1}{r_1}\right]$

$P_1 - P_2 = 0.2 \times 10^3 \times 9.8 - 2 \times 72 \times 10^{-3} \left[\dfrac{1}{7.2 \times 10^{-4}} - \dfrac{1}{14 \times 10^{-4}}\right] = 1960 - 97 = 1863$ Pa

ANGLE OF CONTACT (θ_C)

The angle enclosed between the tangent plane at the liquid surface and the tangent plane at the solid surface at the point of contact inside the liquid is defined as the *angle of contact*. The angle of contact depends the nature of the solid and liquid in contact.

1   **Effect of Temperature on angle of contact**

On increasing temperature surface tension decreases, thus $\cos\theta_c$ increases $\left[\because \cos\theta_c \propto \dfrac{1}{T}\right]$ and θ_c decrease.

So on increasing temperature, θ_c decreases.

1   **Effect of Impurities on angle of contact**

(a) Solute impurities increase surface tension, so $\cos\theta_c$ decreases and angle of contact θ_c increases.

(b) Partially solute impurities decrease surface tension, so angle of contact θ_c decreases.

1   **Effect of Water Proofing Agent**

Angle of contact increases due to water proofing agent. It gets converted acute to obtuse angle.

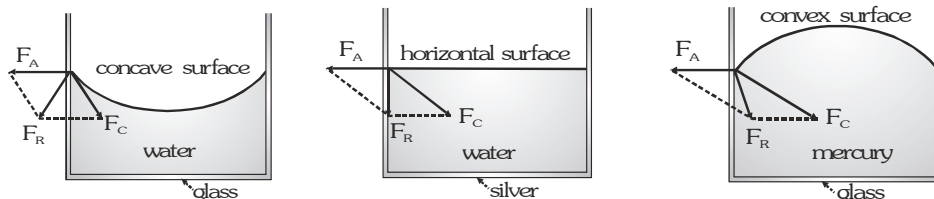### Table of angle of contact of various solid–liquid pairs

Solid - Liquid Pair	θ C
Glass - Normal water	8
Glass - Distilled water	0
Glass - Alcohol	0
Glass - Mercury	135
Paraffin wax - Water	108
Silver - Water	90

Shape of Liquid Surface

When a liquid is brought in contact with a solid surface, the surface of the liquid becomes curved near the place of contact. The shape of the surface (concave or convex) depends upon the relative magnitudes of the cohesive force between the liquid molecules and the adhesive force between the molecules of the liquid and the solid.

The free surface of a liquid which is near the walls of a vessel and which is curved because of surface tension is known as *meniscus*. The cohesive force acts at an angle 45 from liquid surface whereas the adhesive force acts at right angles to the solid surface. The relation between the shape of liquid surface, cohesive/adhesive forces, angle of contact, etc are summarised in the table below :

- Relation between cohesive and adhesive force

|	Concave	Plane	Convex
	$F_A > \dfrac{F_C}{\sqrt{2}}$	$F_A = \dfrac{F_C}{\sqrt{2}}$	$F_A < \dfrac{F_C}{\sqrt{2}}$
• Shape of meniscus	Concave	Plane	Convex
• Angle of contact	$\theta_C < 90$ (Acute angle)	$\theta_C = 90$ (Right angle)	$\theta_C > 90$ (Obtuse angle)
• Shape of liquid drop			
• Level of liquid	Liquid rises up nor falls	Liquid neither rises	Liquid falls
• Wetting property	Liquid wets the solid surface	Liquid does not wet the solid surface	Liquid does not wet the solid surface
• Example	Glass – Water	Silver – Water	Glass – Mercury

CAPILLARY TUBE AND CAPILLARITY

A glass tube with fine bore and open at both ends is known as ***capillary tube***. The property by virtue of which a liquid rise or depress in a capillary tube is known as ***capillarity***. Rise or fall of liquid in tubes of narrow bore (capillary tube) is called ***capillary action***.

Calculation of Capillary Rise
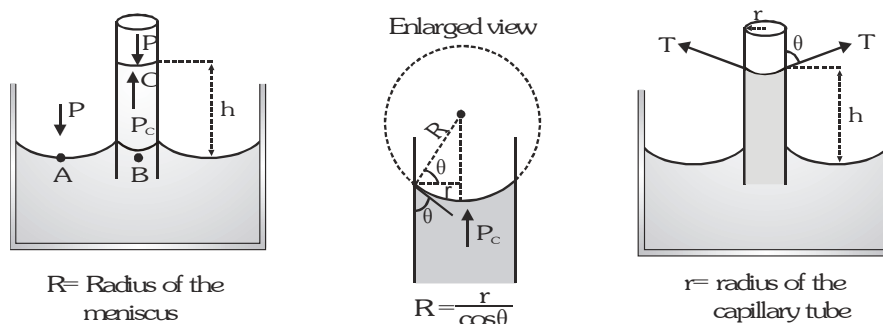
R= Radius of the meniscus

$$R = \frac{r}{\cos\theta}$$

r= radius of the capillary tube

When a capillary tube is first dipped in a liquid as shown in the figure, the liquid climbs up the walls curving the surface. The liquid continues to rise in the capillary tube until the weight of the liquid column becomes equal to force due to surface tension. Let the radius of the meniscus is R and the radius of the capillary tube is r. The angle of contact is θ, surface tension is T, density of liquid is ρ and the liquid rises to a height h.

Now let us consider two points A and B at the same horizontal level as shown. By Pascal's law

$$P_A = P_B \Rightarrow \qquad P_A = P_C + \rho g h \Rightarrow \qquad P_A - P_C = \rho g h \qquad (\because P_B = P_C + \rho g h)$$

Now, the point C is on the curved meniscus which has P_A and P_C as the two pressures on its concave and convex sides respectively.

$$\therefore P_A - P_C = \frac{2T}{R} = \frac{2T}{r / \cos\theta} \Rightarrow \frac{2T}{r / \cos\theta} = \rho\, gh \Rightarrow 2T\cos\theta = r\rho\, gh \Rightarrow h = \frac{2T\cos\theta}{r\rho g}$$
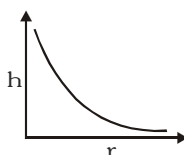
Zurin's Law :

The height of rise of liquid in a capillary tube is inversely proportional to the radius of the capillary tube, if T, θ, ρ and g are constant $h \propto \dfrac{1}{r}$ or *rh* = constant. It implies that liquid will rise more in capillary tube of less radius and vice versa.


GOLDEN KEY POINTS

- For water and clean glass capillary $\theta_c \approx 0°$

- If angle of contact θ_c is acute then $\cos\theta_c$ is positive, so h is positive and liquid rises. If θ_c is obtuse then $\cos\theta_c$ is negative, so h is negative, therefore liquid depresses.

- Rise of liquid in a capillary tube does not obey the law of conservation of energy.

- For two capillaries of radii r_1 and r_2, the capillary rise h_1 and h_2 are such that $\dfrac{h_2}{h_1} = \dfrac{r_1}{r_2}$. The relation

  between h and r is graphically represented as

- Inside a satellite, water will rise upto top level but will not come outside. Radius of curvature(R') increases in such a way that final height h' is reduced and given by $h' = \dfrac{hR}{R'}$. (It is in accordance with Zurin's law).

- If a capillary tube is dipped into a liquid and tilted at an angle α from vertical then the vertical height of liquid column remains same whereas the length of liquid column in the capillary tube increases.
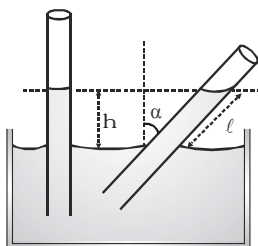
  $$h = \ell\cos\alpha \Rightarrow \ell = \frac{h}{\cos\alpha}$$

- The height 'h' is measured from the bottom of the meniscus. However, there exist some liquid above this line also. If correction of this is applied then the formula will be $\quad T = \dfrac{r\rho g\left[h + \dfrac{1}{3}r\right]}{2\cos\theta}$

- If a hollow sphere of radius r which has a fine hole, drowned in a vessel upto h depth, then liquid will not enter

  upto critical height h, given by $h\rho g = \dfrac{2T\cos\theta}{r}$ [normally $\theta \approx 0°$ therefore $\cos\theta \approx 1$]

Example

Calculate the height to which water will rise in a capillary tube of diameter 1×10^{-3} m. [Given : surface tension of water is 0.072 N m^{-1}, angle of contact is 0, g = 9.8 m s^{-2} and density of water = 1000 kg m^{-3}]

Solution

Height of capillary rise $h = \dfrac{2T\cos\theta}{r\rho g} = \dfrac{2 \times 0.072 \times \cos 0^0}{5 \times 10^{-4} \times 1000 \times 9.8} m = 2.94 \times 10^{-2} m$

Example

Water rises to a height of 20 mm in a capillary. If the radius of the capillary is made one third of its previous value then what is the new value of the capillary rise?

Solution

Since $h = \dfrac{2T\cos\theta}{rdg}$ and for the same liquid and capillaries of difference radii $h_1 r_1 = h_2 r_2$

$\therefore \dfrac{h_2}{h_1} = \dfrac{r_1}{r_2} = \dfrac{1}{(1/3)} = 3$ hence $h_2 = 3h_1 = 3 \times 20$ mm = 60 mm

FLUID STATICS

Matter exists in three states solid, liquid and gas. Liquids and gases are referred to as fluids. Any state of matter that can flow is a fluid. Study of a fluid at rest is called fluid statics or hydrostatics and the study of fluid in motion is called fluid dynamics of hydrodynamics. Both combined are called fluid mechanics.

The intermolecular force in liquids are comparatively weaker than in solids. Therefore, their shape can be changed easily. Thus liquids assume the shape of the container. But their volume (or density) cannot be changed so easily. Liquids are incompressible and have free surface of their own.

The intermolecular forces are weakest in gases, so their shape and size can be changed easily. Gases are compressible and occupy all the space of the container.

DENSITY (ρ)

Mass per unit volume is defined as density. So density at a point of a fluid is represented as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$ Density is a positive scalar quantity.

SI UNIT :   kg/m^3 **CGS UNIT :** g/cc   **Dimensions :** $[ML^{-3}]$

RELATIVE DENSITY

It is defined as the ratio of the density of the given fluid to the density of pure water at 4 C.

$$\text{Relative density (R.D.)} = \frac{\text{density of given liquid}}{\text{density of pure water at 4 C}}$$

Relative density or specific gravity is a unitless and dimensionless positive scalar physical quantity.

Being a dimensionless/unitless quantity R.D. of a substance is same in SI and CGS system.

SPECIFIC GRAVITY

It is defined as the ratio of the specific weight of the given fluid to the specific weight of pure water at 4 C.

$$\text{Specific gravity} = \frac{\text{specific weight of given liquid}}{\text{specific weight of pure water at 4 C (9.81 kN/m}^3\text{)}} = \frac{\rho_\ell \times g}{\rho_w \times g} = \frac{\rho_\ell}{\rho_w} = \text{R.D. of the liquid}$$

Thus specific gravity of a liquid is numerically equal to the relative density of that liquid and for calculation purposes they are used interchangeably.

Example

A hollow metallic sphere has inner and outer radii, respectively, as 5 cm and 10 cm. If the mass of the sphere is 2.5 kg, find (a) density of the material, (b) relative density of the material of the sphere.

Solution

The volume of the material of the sphere is

$$V = \left(\frac{4}{3}\right)\pi\left(r_2^3 - r_1^3\right) = \frac{4}{3} \times 3.14 \times \left[\left(\frac{10}{100}\right)^3 - \left(\frac{5}{100}\right)^3\right] = \frac{4}{3} \times 3.14 \times [0.001 - 0.000125]$$

$$= \frac{4}{3} \quad 3.14 \quad 0.000875 \text{ m}^3 = 0.00367 \text{ m}^3$$

(a) Therefore, density of the material of the sphere is $\rho = \dfrac{M}{V} = \dfrac{2.5}{0.00367}$ kg/m^3 = 681.2 kg/m^3

(b) Relative density of the material of the sphere $\rho_r = \dfrac{681.2}{1000} = 0.6812$

Density of a Mixture of substance in the proportion of mass

Let a number of substances of masses M_1, M_2, M_3 etc., and densities ρ_1, ρ_2, ρ_3 etc. respectively are mixed together. The total mass of the mixture = $M_1 + M_2 + M_3 + ...$

The total volume = $\dfrac{M_1}{\rho_1} + \dfrac{M_2}{\rho_2} + \dfrac{M_3}{\rho_3} + ...$ therefore, the density of the mixture is $\rho = \dfrac{M_1 + M_2 + M_3....}{\dfrac{M_1}{\rho_1} + \dfrac{M_2}{\rho_2} + \dfrac{M_3}{\rho_3} + ...}$

For two substances the density of the mixture $\rho = \dfrac{\rho_1 \rho_2 (M_1 + M_2)}{\rho_1 M_2 + \rho_2 M_1}$

Example

Two immiscible of densities 2.5 g/cm^3 and 0.8 g/cm^3 are taken in the ratio of their masses as 2:3 respectively. Find the average density of the liquid combination.

Solution

Let masses be 2m g & 3m g, then $V = V_1 + V_2 = \left(\dfrac{2m}{2.5} + \dfrac{3m}{0.8} \right)$ cm^3

Total mass = 2m + 3m = 5m g.

Therefore, the average density $\rho_{av} = \dfrac{5m}{V} = \dfrac{5m}{\dfrac{2m}{2.5} + \dfrac{3m}{0.8}} = \dfrac{5}{\dfrac{2}{2.5} + \dfrac{3}{0.8}} = \dfrac{10}{9.1}$ gm/cm^3 = 1.09 gm/cm^3

Density of a mixture of substance in the proportion of volume

Suppose that a number of substances of volume V_1, V_2, V_3 etc. and densities ρ_1, ρ_2, ρ_3 etc. respectively are mixed. The total mass of the mixture is $= \rho_1 V_1 + \rho_2 V_2 + \rho_3 V_3 + ...$

The total volume of the mixture is $= V_1 + V_2 + V_3 + ...$

Therefore, the density of the mixture is $\rho = \dfrac{\rho_1 V_1 + \rho_2 V_2 + \rho_3 V_3}{V_1 + V_2 + V_3 + ...}$

Therefore, for two substances we can write $\rho = \dfrac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2}$

Example

Two miscible liquids of densities 1.2 gm/cc and 1.4 gm/cc are mixed with a proportion ratio of their volumes equal to 3:5 : What is the density of resulting liquid?
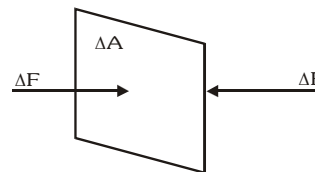
Solution

$\rho_1 = \dfrac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2} \Rightarrow \rho = \dfrac{\rho_1 \dfrac{V_1}{V_2} + \rho_2}{V_1 / V_2 + 1} \Rightarrow \rho = \dfrac{(1.2)(3/5) + 1.4}{(3/5) + 1} = \dfrac{3.6 + 7}{8} \Rightarrow \rho = 1.325$

E

PRESSURE

The pressure P is defined as the magnitude of the normal force acting on a unit surface area.

$$P = \frac{\Delta F}{\Delta A} \quad \text{here } \Delta F = \text{normal force on a surface area } \Delta A$$

The pressure is a scalar quantity. This is because hydrostatic pressure is transmitted equally in all directions when force is applied, which shows that a definite direction is not associated with pressure.

CONSEQUENCES OF PRESSURE

(i)   Railway tracks are laid on large sized wooden or iron sleepers. This is because the weight (force) of the train is spread over a large area of the sleeper. This reduces the pressure acting on the ground and hence prevents the yielding of ground under the weight of the train.

(ii)   A sharp knife is more effective in cutting the objects than a blunt knife. The pressure exerted = Force / area. The sharp knife transmits force over a small area as compared to the blunt knife. Hence the pressure exerted in case of sharp knife is more than in case of blunt knife.

(iii)   A camel walks easily on sand but a man cannot inspite of the fact that a camel is much heavier than man. This is because the area of camel's feet is large as compared to man's feet. So the pressure exerted by camel on the sand is very small as compared to the pressure exerted by man. Due to large pressure, sand under the feet of man yields and hence he cannot walk easily on sand.

TYPES OF PRESSURE

There are three types of pressure

(i) Atmospheric pressure (P_0)   (iii) Gauge pressure (P_{gauge})   (ii) Absolute pressure ($P_{abs.}$)

• **Atmospheric pressure :**

Force exerted by air column on unit cross–section area of sea level called atmospheric pressure (P_o)

$$P_o = \frac{F}{A} = 101.3 \text{ kN/m}^2 \qquad \therefore P_o = 1.013 \times 10^5 \text{ N/m}^2$$
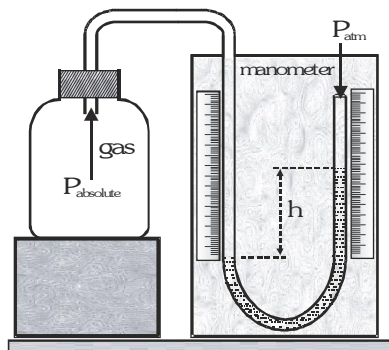
Barometer is used to measure atmospheric pressure.

Which was discovered by **Torricelli**.

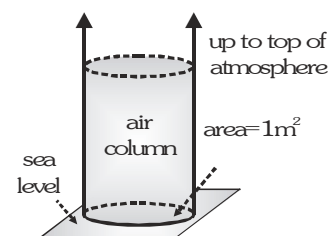Atmospheric pressure varies from place to place and at a particular place from time to time.

• **Gauge Pressure :**

Excess Pressure ($P - P_{atm}$) measured with the help of pressure measuring instrument called Gauge pressure.

$$P_{gauge} = h\rho g \quad \text{or} \quad P_{gauge} \propto h$$

Gauge pressure is always measured with help of **"manometer"**

• **Absolute Pressure :**

Sum of atmospheric and Gauge pressure is called absolute pressure.

$$P_{abs} = P_{atm} + P_{gauge} \Rightarrow P_{abs} = P_o + h\rho g$$

The pressure which we measure in our automobile tyres is gauge pressure.

VARIATION OF PRESSURE WITH DEPTH

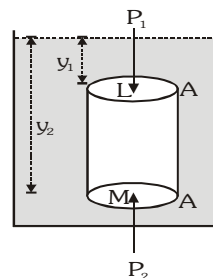(i)   Let pressure at L is P_1 and pressure at M is P_2

Then, $P_2 A = P_1 A + \rho g A (y_2 - y_1) \Rightarrow P_2 = P_1 + \rho g(y_2 - y_1)$

Here pressure gradient $\dfrac{dP}{dy} = \rho g$

(ii)   Pressure is same at two points in the same horizontal level.

As body is in equilibrium, $P_1 A = P_2 A \Rightarrow P_1 = P_2$

Note : Pressure P is independent of shape of container

Example

Assuming that the atmosphere has a uniform density of (1.3 kg/m³) and an effective height of 10 km, find the force exerted on an area of dimensions 100 m   80 m at the bottom of the atmosphere .

Solution

$F = PA = \rho g h A = (1.3) (9.8) (10^4) (100 \times 80) = 12.74 \times 8 \times 10^7 \text{ N} = 1.0192 \times 10^9 \text{ N}$

PRESSURE EXERTED BY A LIQUID (EFFECT OF GRAVITY) :

Consider a vessel containing liquid. As the liquid is in equilibrium, so every volume element of the fluid is also in equilibrium. Consider one volume element in the form of a cylindrical column of liquid of height h and of area of cross section A. The various forces acting on the cylindrical column of liquid are :

(i)   Force, $F_1 = P_1 A$ acting vertically downward on the top face of the column. P_1 is the pressure of the liquid on the top face of the column and is known as atmospheric pressure.

(ii)   Force, $F_2 = P_2 A$ acting vertically upward at the bottom face of the cylindrical column. P_2 is the pressure of the liquid on the bottom face of the column.

(iii) Weight, W = mg of the cylindrical column of the liquid acting vertically downward. Since the cylindrical column of the liquid is in equilibrium, so the net force acting on the column is zero. i.e. $F_1 + W - F_2 = 0$

$\Rightarrow P_1 A + mg - P_2 A = 0 \Rightarrow P_1 A + mg = P_2 A \therefore P_2 = P_1 + \dfrac{mg}{A}$ ...(i)

Now, mass of the cylindrical column of the liquid is,

m = volume   density of the liquid = Area of cross section   height   density = $Ah\rho$

$\therefore$ equation (i) becomes $P_2 = P_1 + \dfrac{Ah\rho g}{A}$, $P_2 = P_1 + h\rho g$ ...(ii)

P_2 is the absolute pressure at depth h below the free surface of the liquid. Equation (ii), shows that the absolute pressure at depth h is greater than the atmospheric pressure (P_1) by an amount equal to $h\rho g$.

Equation (ii) can also be written as $(P_2 - P_1) = h\rho g$ which is the difference of pressure between two points separated by a depth h.
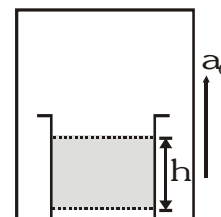
E
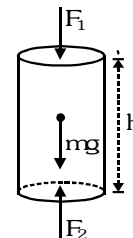
PRESSURE IN CASE OF ACCELERATING FLUID
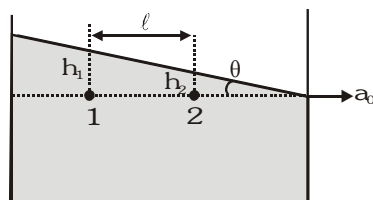
(i) **Liquid placed in elevator :**

When elevator accelerates upward with acceleration a_0 then pressure in the fluid, at depth 'h' may be given by,

$$P = h\rho[g + a_0]$$

(ii) **Free surface of liquid in case of horizontal acceleration :**

$$\tan\theta = \frac{ma_0}{mg} = \frac{a_0}{g}$$

If P_1 and P_2 are pressures at point 1 & 2 then $P_1 - P_2 = \rho g(h_1 - h_2) = \rho g \ell \tan\theta = \rho \ell a_0$

(iii) **Rotating Vessel**

Consider a cylindrical vessel, rotating at constant angular velocity about its axis. If it contains fluid then after an initial irregular shape, it will rotate with the tank as a rigid body. The acceleration of fluid particles located at a distance r from the axis of rotation will be equal to $\omega^2 r$, and the direction of the acceleration is toward the axis of rotation as shown in the figure. The fluid particles will be undergoing circular motion.

Lets consider a small horizontal cylinder of length dr and cross-sectional area A located y below the free surface of the fluid and r from the axis. This cylinder is accelerating in ground frame with acceleration $\omega^2 r$ towards the axis hence the net horizontal force acting on it should be equal to the product of mass (dm) and acceleration.

$$dm = Adr\rho$$

$$P_2 A - P_1 A = (Adr\rho)\omega^2 r$$

If we say that the left face of the cylinder is y below the free surface of the fluid then the right surface is y + dy below the surface of liquid. Thus $P_2 - P_1 = \rho g dy$

Thus solving we get $\qquad \dfrac{dy}{dr} = \dfrac{r\omega^2}{g}$

and, therefore, the equation for surfaces of constant pressure is $\quad y = \dfrac{\omega^2 r^2}{2g}$ + constant

This equation means that these surface of constant pressure are parabolic as shown in figure.
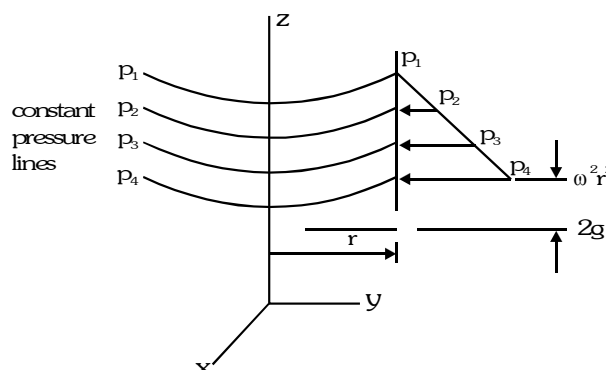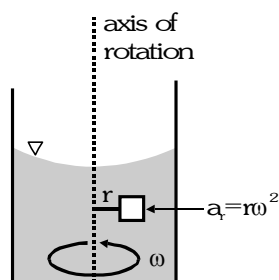
The pressure varies with the distance from the axis of rotation, but at a fixed radius, the pressure varies hydrostatically in the vertical direction as shown in figure.

Example

An open water tanker moving on a horizontal straight road has a cubical block of cork floating over its surface. If the tanker has an acceleration of 'a' as shown, the acceleration of the cork w.r.t. container is (ignore viscosity)
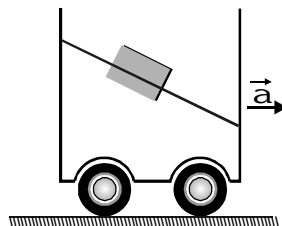
Solution
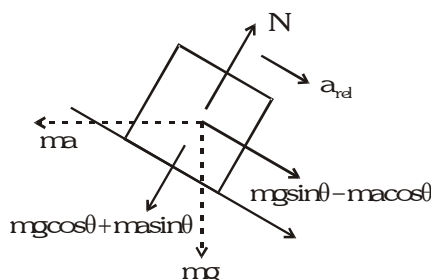
ma_{rel} = mgsinθ – macosθ   but for water surface tan θ = a/g $\Rightarrow a_{rel}$ = 0
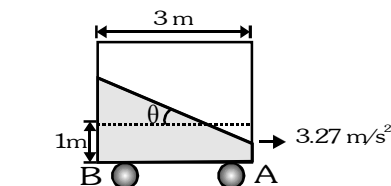
Example

An open rectangular tank 1.5 m wide, 2 m deep and 2m long is half filled with water. It is accelerated horizontally at 3.27 m/s² in the direction of its length. Determine the depth of water at each end of tank. [g = 9.81 m/s²]

Solution

Here $\tan\theta = \dfrac{a}{g} = \dfrac{1}{3}$, depth at corner 'A' = 1– 1.5 tan θ = 0.5 m

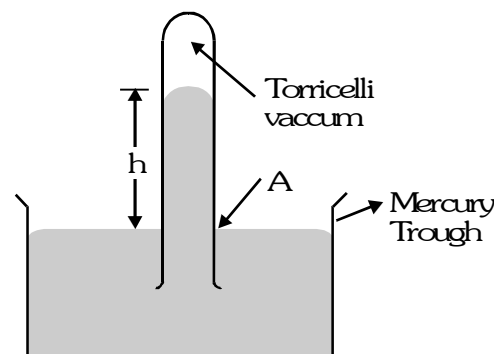depth at corner 'B' = 1 + 1.5 tan θ = 1.5 m

MEASUREMENT OF ATMOSPHERIC PRESSURE

1.   **Mercury Barometer :**

To measure the atmospheric pressure experimentally, torricelli invented a mercury barometer in 1643.

$$p_a = h\rho g$$

The pressure exerted by a mercury column of 1mm high is called 1 Torr.

1 Torr = 1 mm of mercury column

OPEN TUBE MANOMETER :

Open-tube manometer is used to measure the pressure gauge. When equilibrium is reached, the pressure at the bottom of left limb is equal to the pressure at the bottom of right limb.
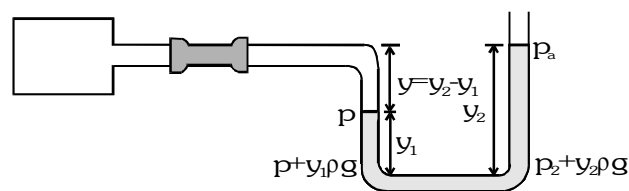
i.e. $p + y_1 \rho g = p_a + y_2 \rho g$

$p - p_a = \rho g (y_2 - y_1) = \rho g y$
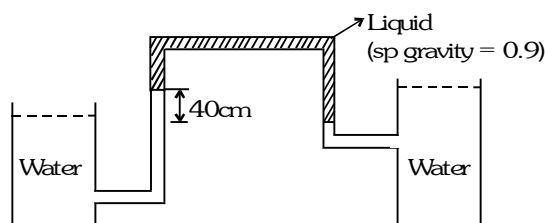
$p - p_a = \rho g (y_2 - y_1) = \rho g y$

p = absolute pressure, $p - p_a$ = gauge pressure.

Thus, knowing y and ρ (density of liquid), we can measure the gauge pressure.

Example

The manometer shown below is used to measure the difference in water level between the two tanks. Calculate this difference for the conditions indicated

Liquid (sp gravity = 0.9)
40cm
Water   Water

Solution

$p_a + h_1 \rho g - 40\rho_1 g + 40\rho g = p_a + h_2 \rho g$

$h_2 \rho g - h_1 \rho g = 40 \rho g - 40 \rho_1 g$ as $\rho_1 = 0.9\rho$

$(h_2 - h_1) \rho g = 40\rho g - 36\rho g$

$h_2 - h_1 = 4$ cm
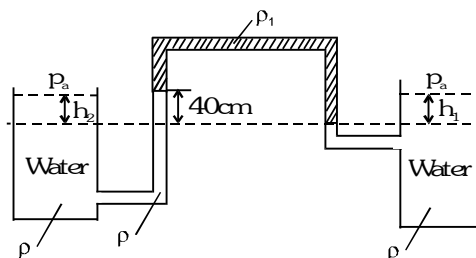
WATER BAROMETER

Let us suppose water is used in the barometer instead of mercury. $h\rho g = 1.013 \times 10^5$ or $h = \dfrac{1.013 \times 10^5}{\rho g}$

The height of the water column in the tube will be 10.3 m. Such a long tube cannot be managed easily, thus water barometer is not feasible.

Example

In a given U-tube (open at one-end) find out relation between p and p_a.

Given $d_2 = 2 \times 13.6$ gm/cm³ $d_1 = 13.6$ gm/cm³

Solution

Pressure in a liquid at same level is same i.e. at A – A–,

$p_a + d_2 yg + x d_1 g = p$

In C.G.S. $p_a + 13.6 \times 2 \times 25 \times g + 13.6 \times 26 \times g = p$

$p_a + 13.6 \times g [50 + 26] = p$

$\Rightarrow 2p_a = p$ as $p_a = 13.6 \times g \times 76$

Example

Find out pressure at points A and B. Also find angle 'θ'.
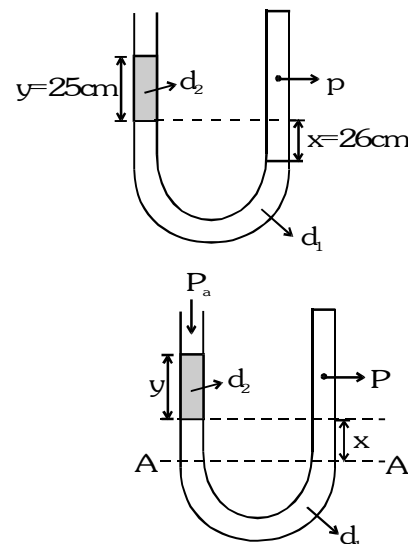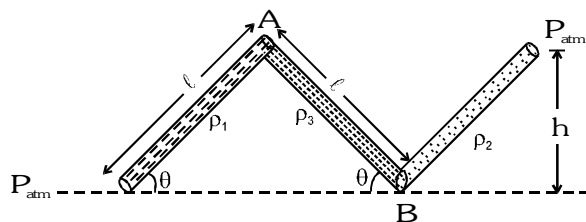

Solution

Pressure at A – $P_A = P_{atm} - \rho_1\, g\ell\, \sin\theta$

Pressure at B $P_B = P_{atm} + \rho_2\, gh\, \theta$

But P_B is also equal to $P_B = P_A + \rho_3\, g\ell\, \sin\theta$

Hence - $P_{atm} + \rho_2\, gh = P_A + \rho_3\, g\ell\, \sin\theta$

$P_{atm} + \rho_2\, gh = P_{atm} - \rho_1\, g\ell\, \sin\theta + \rho_3\, g\ell\, \sin\theta$

$$\sin\theta = \frac{\rho_2 h}{(\rho_3 - \rho_1)\ell}\,.$$

Example

In the given figure, the container slides down with acceleration 'a' on an incline of angle 'θ'. Liquid is stationary with respect to container. Find out

(i) Angle made by surface of liquid with horizontal plane.

(ii) Angle if $a = g\,\sin\theta$.
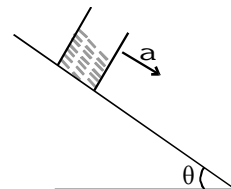
Solution

Consider a fluid particle on surface. The forces acting on it are shown in figure.
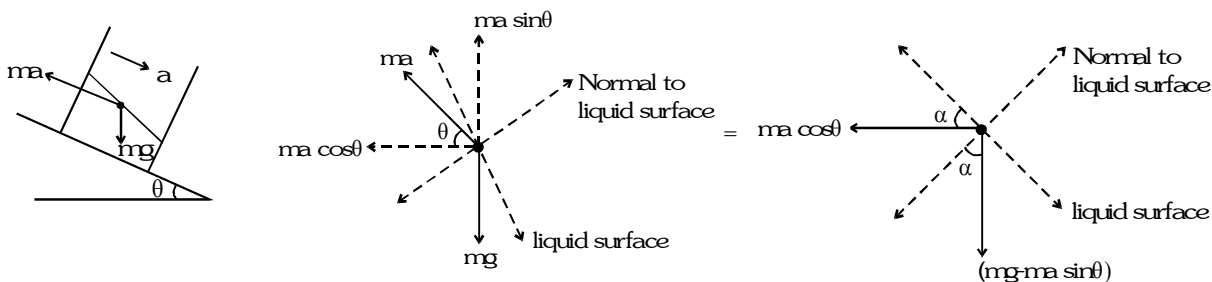
Resultant force acting on liquid surface, will always normal to it $\tan\alpha = \dfrac{ma\cos\theta}{mg - ma\sin\theta} = \dfrac{a\cos\theta}{(g - a\sin\theta)}$

Thus angle of liquid surface with the horizontal is equal to $\alpha = \tan^{-1}\dfrac{a\cos\theta}{(g - a\sin\theta)}$

(ii) If $a = g\,\sin\theta$, then $\alpha = \tan^{-1}\left(\dfrac{a\cos\theta}{g - g\sin^2\theta}\right)$ $= \tan^{-1}\dfrac{g\sin\theta\cos\theta}{g\cos^2\theta}$

$= \tan^{-1}(\tan\theta)$

$\Rightarrow \alpha = \theta$

Example

An L shaped glass tube is kept inside a bus that is moving with constant acceleration. During the motion, the level of the liquid in the left arm is at 12 cm whereas in the right arm, it is at 8 cm when the orientation of the tube is as shown. Assuming that the diameter of the tube is much smaller than levels of the liquid and neglecting effect of surface tension, acceleration of the bus find the (g = 10 m/s²).
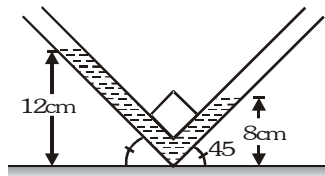
Solution

$$\tan\theta = \frac{a}{g} = \frac{h_2 - h_1}{h_2 \tan 45° + h_1 \tan 45°} = \frac{4\,cm}{20\,cm} \Rightarrow a = 2\ m/s^2$$

FORCE ON SIDE WALL OF VESSEL

Force on the side wall of the vessel can not be directly determined as a different depths pressures are different. To find this we consider a strip of width dx at a depth x from the surface of the liquid as shown in figure, and on this strip the force due to the liquid is given as : $dF = x\rho g\ bdx$
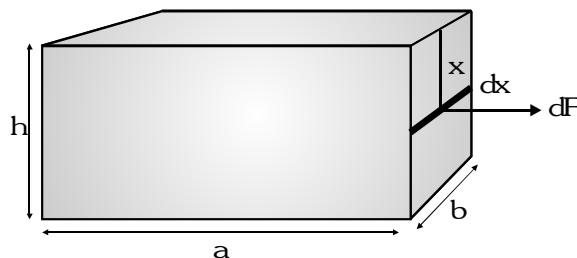
This force is acting in the direction normal to the side wall. Net force can be evaluated by integrating equation

$$F = \int dF = \int_0^h x\rho gbdx \qquad F = \frac{\rho gbh^2}{2}$$

Average pressure on side wall

The absolute pressure on the side wall cannot be evaluated because at different depths on this wall pressure is different. The average pressure on the wall can be given as :

$$\langle P \rangle_{av} = \frac{F}{bh} = \frac{1}{2}\frac{\rho gbh^2}{bh} = \frac{1}{2}\rho gh$$

Equation shows that the average pressure on side vertical wall is half of the net pressure at the bottom of the vessel.

TORQUE ON THE SIDE WALL DUE TO FLUID PRESSURE

As shown in figure, due to the force dF, the side wall experiences a torque about the bottom edge of the side which is given as $\quad d\tau = dF \times (h - x) = x\rho gb\, dx(h - x)$

This net force is $\quad \tau = \int d\tau = \int_0^h \rho gb\left(hx - x^2\right)dx$

$$= \rho gb\left[\frac{h^3}{2} - \frac{h^3}{3}\right] = \frac{1}{6}\rho gbh^3$$

Example

Water and liquid is filled up behind a square wall of side ℓ. Find out
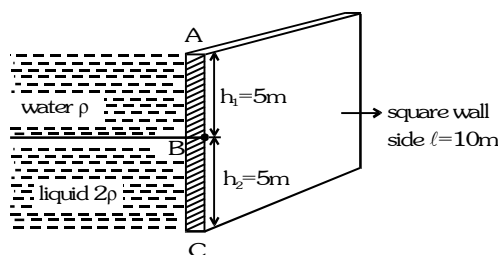

(a) Pressures at A, B and C   (b) Forces in part AB and BC

(c) Total force and point of application of force. (Neglect atmosphere pressure in every calculation)

Solution

(a)   As there is no liquid above 'A'

So pressure at A, $p_A = 0$

Pressure at B, $p_B = \rho gh_1$

Pressure at C, $p_C = \rho gh_1 + 2\rho gh_2$

(b)   Force at A = 0

Take a strip of width 'dx' at a depth 'x' in part AB.

Pressure is equal to ρgx.

Force on strip = pressure   area

$dF = \rho gx\, \ell dx$

Total force upto B : $F = \int_0^{h_1} \rho gx\ell dx = \frac{\rho gx\ell h_1^2}{2} = \frac{1000 \times 10 \times 10 \times 5 \times 5}{2} = 1.25 \times 10^6\, N$

In part BC for force take a elementary strip of width dx in portion BC. Pressure is equal to $= \rho gh_1 + 2\rho g(x - h_1)$

Force on elementary strip = pressure   area $dF = [\rho gh_1 + 2\rho g(x - h_1)]\, \ell\, dx$

Total force on part BC $\quad F = \int_{h_1}^{\ell} [\rho gh_1 + 2\rho g(x - h_1)]\ell\, dx$

$$= \left[\rho gh_1x + 2\rho g\left[\frac{x^2}{2} - h_1x\right]\right]_{h_1}^{\ell} \ell$$

$$= \rho gh_1h_2\, \ell + 2\rho g\ell\left[\frac{\ell^2 - h_1^2}{2} - h_1\ell + h_1^2\right]$$
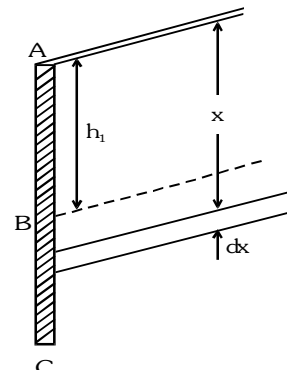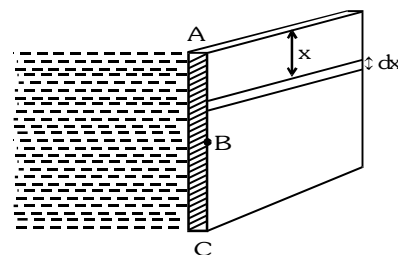
$$= \rho g h_1 h_2 \ell + \frac{2\rho g \ell}{2} [\ell^2 + h_1^2 - 2h_1\ell] = \rho g h_1 h_2 \ell + \rho g \ell (\ell - h_1)^2$$

$$= \rho g h_2 \ell [h_1 + h_2] = \rho g h_2 \ell^2 = 1000 \times 10 \times 5 \times 10 \times 10 = 5 \times 10^6 \text{ N}$$

(c)   Total force $= 5 \times 10^6 + 1.25 \times 10^6 = 6.25 \times 10^6 \text{ N}$

Taking torque about A

Total torque of force in AB $= \int dF \cdot x = \int_0^{h_1} \rho g x \ell dx \cdot x$

$$= \left[\frac{\rho g \ell x^3}{3} \right]_0^{h_1} = \frac{\rho g \ell h_1^3}{3} = \frac{1000 \times 10 \times 10 \times 125}{3} = \frac{1.25 \times 10^7}{3} N - m$$

Total torque of force in BC $= \int dF \cdot x$

On solving we get $= \rho g h_1 h_2 \ell [h_1 + \frac{h_2}{2}] + \rho g h_2^2 \ell [h_1 + \frac{2h_2}{3}]$

$$= 1000 \times 10 \times 5 \times 5 \times 10 [5 + 2.5] + 1000 \times 10 \times 25 \times 10 [5 + \frac{10}{3}]$$

$$= 2.5 \times 7.5 \times 10^6 + \frac{62.5}{3} \times 10^6 = \frac{118.75}{3} \times 10^6$$

Total torque $= \frac{11.875 \times 10^7}{3} + \frac{1.25 \times 10^7}{3} = \frac{13.125 \times 10^7}{3}$

Total torque = total force $\times$ distance of point of application of force from top = $F.x_p$

$$6.25 \times 10^6 x_p = \frac{13.125 \times 10^7}{3}$$

$$x_p = 7m$$

PASCAL's LAW

If the pressure in a liquid is changed at a particular point, the change is transmitted to the entire liquid without being diminished in magnitude. Pascal's law is stated in following ways –

- The pressure in a fluid at rest is same at all the points if gravity is ignored.

- A liquid exerts equal pressures in all directions.

- If the pressure in an enclosed fluid is changed at a particular point, the change is transmitted to every point of the fluid and to the walls of the container without being diminished in magnitude.
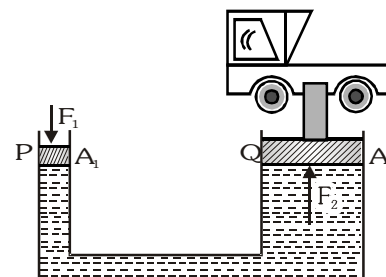
Applications of pascal's law hydraulic jacks, lifts, presses, brakes, etc

For the hydraulic lift

Pressure applied $= \dfrac{F_1}{A_1}$

$\therefore$ Pressure transmitted $= \dfrac{F_2}{A_2}$ $\because$ $\dfrac{F_1}{A_1} = \dfrac{F_2}{A_2}$

$\therefore$ Upward force on A_2 is $F_2 = \dfrac{F_1}{A_1} \times A_2 = \dfrac{A_2}{A_1} \times F_1$

Ex. A vertical U–tube of uniform cross–section contains mercury in both arms. A glycerine (relative density = 1.3) column of length 10 cm is introduced into one of the arms. Oil of density 800 kg m^{-3} is poured into the other arm until the upper surface of the oil and glycerine are at the same horizontal level. Find the length of the column. Density of mercury is 13.6×10^3 kgm^{-3}

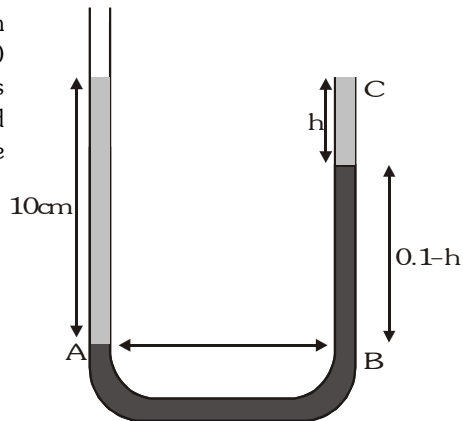**Sol.** Pressure at A and B must be same

Pressure at A = $P_0 + 0.1 \times (1.3 \times 1000) \times g$

Pressure at B = $P_0 + h \times 800 \times g + (0.1 - h) \times 13.6 \times 1000\, g$

$\Rightarrow 0.1 \times 1300 = 800 \, h + (0.1 - h) \times 13600$

$\Rightarrow h = 0.096 \text{ m} = 9.6 \text{ cm}$

BUOYANCY AND ARCHIMEDE'S PRINCIPLE

- **Buoyant Force** : If a body is partially or wholly immersed in a fluid, it experiences an upward force due to the fluid surrounding it. This phenomenon of force exerted by fluid on the body is called *buoyancy* and force is called *buoyant force* or *upthrust*.

- **Archimede's Principle** : It states that the buoyant force on a body that is partially or totally immersed in a liquid is equal to the weight of the fluid displaced by it.

Now consider a body immersed in a liquid of density σ.

Top surface of the body experiences a downward force
$$F_1 = AP_1 = A[h_1\sigma g + P_0] \quad ...(i)$$
Lower face of the body will experiences a upward force
$$F_2 = AP_2 = A[h_2\sigma g + P_0] \quad ...(ii)$$
As $h_2 > h_1$. So F_2 is greater than F_1

So net upward force : $F = F_2 - F_1 = A\sigma g[h_2 - h_1]$
$$\therefore \quad F = A\sigma g L = V\sigma g \; [\because \; V = AL]$$
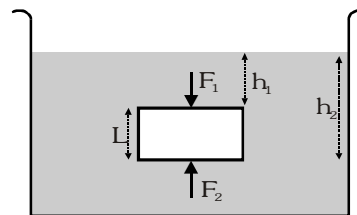
FLOATATION

When a body of density (ρ) and volume (V) is completely immersed in a liquid of density (σ), the forces acting on the body are :

(i) Weight of the body W = Mg = Vρg (directed vertically downwards through C.G. of the body).

(ii) Buoyant force or Upthrust Th = Vσg (directed vertically upwards through C.B.).

The apparent weight W_{App} is equal to W – Th.

The following three cases are possible :

Case I Density of the body (ρ) is greater than that of liquid (σ)

In this case if $\rho > \sigma$ then W > Th

So the body will sink to the bottom of the liquid.

$$W_{App} = W - Th = V\rho g - V\sigma g = V\rho g \, (1 - \sigma/\rho) = W \, (1 - \sigma/\rho).$$

Case II Density of the body is equal to the density of liquid ($\rho = \sigma$)

In this case if $\rho = \sigma$ then W = Th

So the body will float fully submerged in the liquid. It will be in neutral equilibrium.

$$W_{App} = W - Th = 0$$

E

Case III   Density of the body is lesser than that of liquid ($\rho < \sigma$)

In this case   if $\rho < \sigma$ then $W < Th$

So the body will float partially submerged in the liquid. In this case the body will move up and the volume of liquid displaced by the body (V_{in}) will be less than the volume of body (V). So as to make Th equal to W

$$\therefore W_{App} = W - Th = 0$$

The above three cases constitute the **law of flotation** which states that a body will float in a liquid if weight of the liquid displaced by the immersed part of the body is at least equal to the weight of the body.

## Rotatory – Equilibrium in Floatation :

When a floating body is slightly tilted from equilibrium position, the centre of buoyancy B shifts. The vertical line passing through the new centre of buoyancy B' and initial vertical line meet at a point M called meta – centre. If the metacentre M is above the centre of gravity the couple due to forces at G (weight of body W) and at B' (upthrust) tends to bring the body back to its original position (figure) . So for rotational equilibrium of floating body the meta–centre must always be higher than the centre of gravity of the body.
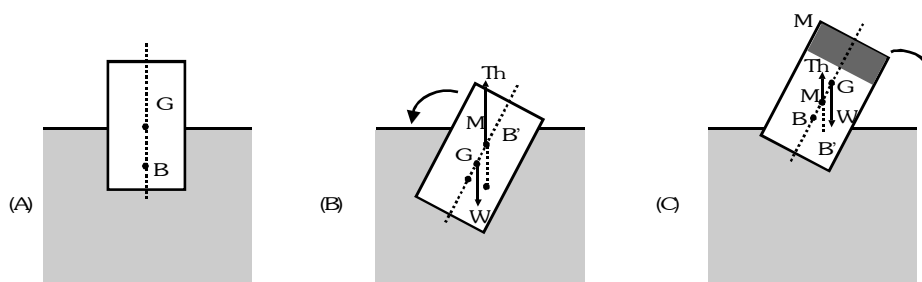
However, if meta–centre goes below centre of gravity, the couple due to forces at G and B' tends to topple the floating body. This is why a wooden log cannot be made to float vertical in water or a boat is likely to capsize if the sitting passengers stand on it. In these situations centre of gravity becomes higher than meta centre and so the body will topple if slightly tilted.

Example

A cubical block of wood of edge 3 cm floats in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. Find the maximum weight that can be put on the block without wetting it. Density of wood = 800 kg/m³ and spring constant of the spring = 50 N/m. Take g = 10 m/s²
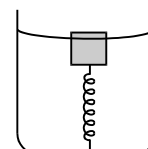
Solution

The specific gravity of the block = 0.8. Hence the height inside water = 3 cm $\times$ 0.8 = 2.4 cm. The height outside water = 3 cm – 2.4 = 0.6 cm. Suppose the maximum weight that can be put without wetting it is W. The block in this case is completely immersed in the water.

The volume of the displaced water= volume of the block = 27×10^{-6} m³.

Hence, the force of buoyancy = (27×10^{-6} m³) $\times$ 1(1000 kg/m3) $\times$ (10 m/s²) = 0.27 N.

The spring is compressed by 0.6 cm and hence the upward force exerted by the spring
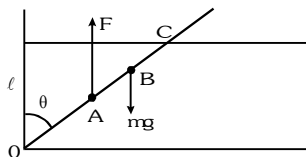
$= 50$ N/m $\times$ 0.6 cm = 0.3 N.

The force of buoyancy and the spring force taken together balance the weight of the block plus the weight W put on the block. The weight of the block is

$W' = (27 \times 10^{-6}$ m) $\times$ (800 kg/m³) $\times$ (10 m/s²) = 0.22 N.

Thus,   W = 0.27 N + 0.3 N – 0.22 N = 0.35 N.

Example

A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank as shown in figure.

The tank is filled with water up to a height of 0.5 m. The specific gravity of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position. (Exclude the case θ = 0).

Solution

The forces acting on the plank are shown in the figure. The height of water level is ℓ = 0.5 m. The length of the plank is 1.0 m = 2ℓ. The weight of the plank acts through the centre B of the plank. We have OB = ℓ. The buoyant force F acts through the point A which is the middle point of the dipped part OC of the plank.

We have $\qquad OA = \dfrac{OC}{2} = \dfrac{\ell}{2\cos\theta}$.

Let the mass per unit length of the plank be ρ. Its weight mg = $2\ell\rho g$.

The mass of the part OC of the plank = $\left(\dfrac{\ell}{\cos\theta}\right)\rho$.

The mass of water displaced = $\dfrac{1}{0.5}\dfrac{\ell}{\cos\theta}\rho = \dfrac{2\ell\rho}{\cos\theta}$.

The buoyant force F is, therefore, $F = \dfrac{2\ell\rho g}{\cos\theta}$.

Now, for equilibrium, the torque of mg about O should balance the torque of F about O.

So, mg (OB) sinθ = F(OA) sinθ or, $(2\ell\rho)\ell = \left(\dfrac{2\ell\rho}{\cos\theta}\right)\left(\dfrac{\ell}{2\cos\theta}\right)$ or, $\cos^2\theta = \dfrac{1}{2}$ or, $\cos\theta = \dfrac{1}{\sqrt{2}}$, or, θ = 45.

GOLDEN KEY POINTS

- Buoyant force act vertically upward through the centre of gravity (C.G.) of the displaced fluid. This point is called centre of buoyancy (C.B.). Thus centre of buoyancy is the point through which the force of buoyancy is supposed to act.

- Buoyant force or upthrust does not depend upon the characteristics of the body such as its mass, size, density, etc. But it depends upon the volume of the body inside the liquid.

- It depends upon the nature of the fluid as it is proportional to the density of the fluid.

  This is the reason that upthrust on a fully submerged body is more in sea water than in pure water

- It depends upon the *effective acceleration*.

  If a lift is accelerated downwards with acceleration a (a< g) then $\qquad$ Th = $V_{in}\,\sigma\,(g - a)$

  If a lift is accelerated downwards with a = g then $\qquad$ Th = $V_{in}\,\sigma\,(g - a) = 0$

  If a lift is accelerated upward with accelaration a then $\qquad$ Th = $V_{in}\,\sigma\,(g + a)$

- If a body is weighed in air (W_A), in water (W_W) and in a oil (W_O), then

$$\text{Specific gravity of oil} = \dfrac{\text{loss of weight in oil}}{\text{loss of weight in water}} = \dfrac{W_A - W_O}{W_A - W_W}$$

Example

A body weighs 160 g in air, 130 g in water and 136 g in oil. What is the specific gravity of oil?

Solution

$$\text{Specific gravity of oil} = \frac{\text{loss of weight in oil}}{\text{loss of weight in water}} = \frac{160-136}{160-130} = \frac{24}{30} = \frac{8}{10} = 0.8$$

Example

An iceberg is floating partially immersed in sea–water. The density of sea–water is 1.03 gm/cm³ and that of ice is 0.92 gm/cm³. What is the fraction of the total volume of the iceberg above the level of sea–water ?

Solution

In case of flotation weight = upthrust i.e.

$$mg = V_{in}\sigma g \Rightarrow V\rho g = V_{in}\sigma g \therefore V_{in} = \frac{\rho}{\sigma}V \text{ so } V_{out} = V - V_{in} = V\left[1 - \frac{\rho}{\sigma}\right]$$

$$\Rightarrow f\, f_{out} = \frac{V_{out}}{V} = \left[1 - \frac{\rho}{\sigma}\right]\left[1 - \frac{0.92}{1.03}\right] = \frac{0.11}{1.03} = 0.106 \Rightarrow f_{out} = 10.6\ \%$$

Example

A rubber ball of mass 10 gm and volume 15 cm³ is dipped in water to a depth of 10m. Assuming density of water uniform throughout the depth if it is released from rest. Find (take g = 980 cm/s²)

(a) the acceleration of the ball, and   (b) the time taken by it to reach the surface.

Solution

The maximum buoyant force on the ball is $F_B = V\rho_w g = 15 \times 1 \times 980$ dyne = 14700 dyne.

The weight of the ball is mg = 10 $\times$ g = 10 $\times$ 980 = 9800 dyne

The net upward force, F = (15 $\times$ 980 – 10 $\times$ 980) dyne = 5 $\times$ 980 dyne = 4900 dyne

(a) Therefore, acceleration of the ball upward $a = \dfrac{F}{m} = \dfrac{5 \times 980}{10}$ = 490 cm/s² = 4.9 m/s²

(b) Time taken by it reach the surface is $t = \sqrt{\dfrac{2h}{a}} = \sqrt{\dfrac{2 \times 10}{4.9}}$ s = 2.02 s

FLUID DYNAMICS

When a fluid moves in such a way that there are relative motions among the fluid particles, the fluid is said to be flowing.

TYPES OF FLUID FLOW : Fluid flow can be classified as :

l **Steady and Unsteady Flow**

Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure and density at a point do not change with time. In an *unsteady flow*, the velocity, pressure and density at a point in the flow varies with time.

l **Streamline Flow**

In steady flow all the particles passing through a given point follow the same path and hence a unique line of flow. This line or path is called a *streamline*. Streamlines do not intersect each other because if they intersect each other the particle can move in either direction at the point of intersection and flow cannot be steady.

l **Laminar and Turbulent Flow**

Laminar flow is the flow in which the fluid particles move along well–defined streamlines which are straight and parallel. In laminar flow the velocities at different points in the fluid may have different magnitudes, but there directions are parallel. Thus the particles move in laminar or layers gliding smoothly over the adjacent layer.

Turbulent flow is an irregular flow in which the particles can move in zig–zag way due to which eddies formation take place which are responsible for high energy losses.

l **Compressible and Incompressible Flow**

In *compressible flow* the density of fluid varies from point to point i.e. the density is not constant for the fluid whereas in *incompressible flow* the density of the fluid remains constant throughout. Liquids are generally incompressible while gases are compressible.

l **Rotational and Irrotational Flow**

Rotational flow is the flow in which the fluid particles while flowing along path–lines also rotate about their own axis. In *irrotational flow* particles do not rotate about their axis. So they have no net angular velocity.

EQUATION OF CONTINUITY

The continuity equation is the mathematical expression of the *law of conservation of mass in fluid dynamics*.
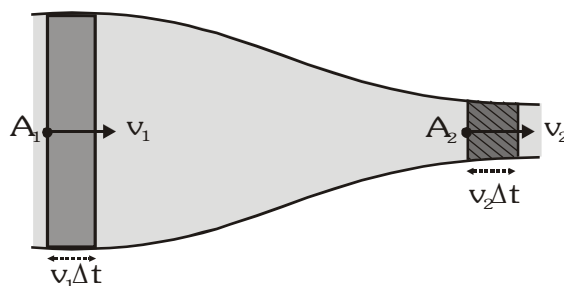
In the steady flow the mass of fluid entering into a tube of flow in a particular time interval is equal to the mass of fluid leaving the tube.

$$\frac{m_1}{\Delta t} = \frac{m_2}{\Delta t} \Rightarrow \rho_1 A_1 v_1 = \rho_2 A_2 v_2 \ (\because \rho_1 = \rho_2) \Rightarrow A_1 v_1 = A_2 v_2 \quad \therefore \ Av = \text{constant}$$

(Here ρ = density of fluid, v = velocity of fluid, A = Area of cross–section of tube)

Therefore the velocity of liquid is smaller in the wider parts of a tube and larger in the narrower parts.

BERNOULLI'S THEOREM

Bernoulli's equation is mathematical expression of the law of mechanical energy conservation in fluid dynamics. Bernoullis theorem is applied to the ideal fluids.

Characteristics of an ideal fluid are :

(i) The fluid is incompressible.

(ii) The fluid is non–viscous.

(iii) The fluid flow is steady.

(iv) The fluid flow is irrotational.

Every point in an ideal fluid flow is associated with three kinds of energies :

Kinetic Energy

If a liquid of mass (m) and volume (V) is flowing with velocity (v) then   Kinetic Energy $= \dfrac{1}{2} mv^2$ and kinetic energy

per unit volume $= \dfrac{\text{Kinetic Energy}}{\text{volume}} = \dfrac{1}{2}\dfrac{m}{V}v^2 = \dfrac{1}{2}\rho v^2$

Potential Energy

If a liquid of mass (m) and volume (V) is at height (h) from the surface of the earth then its

Potential Energy $= mgh$ and potential energy per unit volume $= \dfrac{\text{Potential Energy}}{\text{volume}} = \dfrac{m}{V}gh = \rho gh$

Pressure Energy

If P is the pressure on area A of a liquid and the liquid moves through a distance (ℓ) due to this pressure then

Pressure energy = Work done = force x displacement = pressure x area x displacement $= PA\ell = PV$

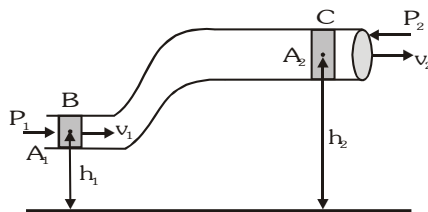$$[\because A\ell = \text{volume } V]$$

Pressure energy per unit volume $= \dfrac{\text{Pressure energy}}{\text{volume}} = P$

Theorem

According to Bernoulli's Theorem , in case of steady flow of incompressible and non–viscous fluid through a tube of non–uniform cross–section, the sum of the pressure, the potential energy per unit volume and the

kinetic energy per unit volume is same at every point in the tube, i.e., $P + \rho gh + \dfrac{1}{2}\rho v^2 = $ constant.

Consider a liquid flowing steadily through a tube of non–uniform area of cross–section as shown in figure. If P_1 and P_2 are the pressures at the two ends of the tube respectively, work done in pushing the volume ΔV of incompressible liquid from point B to C through the tube

$$W = P\,(\Delta V) = (P_1 - P_2)\Delta V \qquad\qquad …(i)$$

This work is used by the liquid in two ways :

(i)   In changing the potential energy of mass Δm (in the volume ΔV)   from

$$\Delta mgh_1 \text{ to } \Delta mgh_2 \text{ i.e., } \Delta U = \Delta mg\,(h_2 - h_1) \qquad\qquad …(ii)$$

(ii)   In changing the kinetic energy from $\dfrac{1}{2}\Delta m v_1^2$ to $\dfrac{1}{2}\Delta m v_2^2$, i.e. $\Delta K = \dfrac{1}{2}\Delta m \left(v_2^2 - v_1^2\right)$

Now as the liquid is non–viscous, by conservation of mechanical energy,

$$W = \Delta U + \Delta K \text{ i.e., } \left(P_1 - P_2\right)\Delta V = \Delta mg\left(h_2 - h_1\right) + \dfrac{1}{2}\Delta m\left(v_2^2 - v_1^2\right)$$
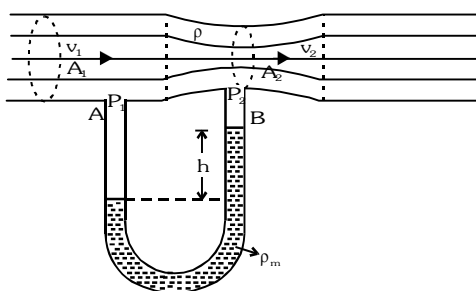
$$P_1 - P_2 = \rho g(h_2 - h_1) + \dfrac{1}{2}\rho\left(v_2^2 - v_1^2\right) \text{ [as } \rho = \Delta m / \Delta V \text{]}$$

$$P_1 + \rho g h_1 + \dfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \dfrac{1}{2}\rho v_2^2 \Rightarrow P + \rho gh + \dfrac{1}{2}\rho v^2 = \text{ constant}$$

This equation is the Bernoulli's equation and represents conservation of mechanical energy in case of moving fluids.

VENTURIMETER

It is a gauge put on a flow pipe to measure the speed of flow of a liquid shown in figure.

Let the liquid of density ρ be flowing through a pipe of area of cross section A_1. Let A_2 be the area of cross section at the throat and a manometer is attached as shown in the figure. Let v_1 and P_1 be the velocity of the flow and pressure at point A, v_2 and P_2 be the corresponding quantities at point B.

Using Bernoulli's theorem :

$$\dfrac{P_1}{\rho} + gh_1 + \dfrac{1}{2}v_1^2 = \dfrac{P_2}{\rho} + gh_2 + \dfrac{1}{2}v_2^2$$

we get $\dfrac{P_1}{\rho} + gh + \dfrac{1}{2}v_1^2 = \dfrac{P_2}{\rho} + gh + \dfrac{1}{2}v_2^2$ (Since $h_1 = h_2 = h$) or $(P_1 - P_2) = \dfrac{1}{2}\rho(v_2^{\ 2} - v_1^{\ 2})$ ....(i)

According to continuity equation, $A_1 v_1 = A_2 v_2$ or $v_2 = \left(\dfrac{A_1}{A_2}\right) v_1$

Substituting the value of v_2 in equation (i)

we have $\left(P_1 - P_2\right) = \dfrac{1}{2}\rho = \left[\left(\dfrac{A_1}{A_2}\right)^2 v_1^2 - v_1^2 \right]\dfrac{1}{2}\rho v_1^2\left[\left(\dfrac{A_1}{A_2}\right)^2 - 1\right]$

Since $A_1 > A_2$, therefore, $P_1 > P_2$ or $v_1^2 = \dfrac{2(P_1 - P_2)}{\rho\left[\left(\dfrac{A_1}{A_2}\right)^2 - 1\right]} = \dfrac{2A_2^2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}$

where $(P_1 - P_2) = \rho_m gh$ and h is the difference in heights of the liquid levels in the two tubes.

$$v_1 = \sqrt{\dfrac{2\rho_m\, gh}{\rho\left[\left(\dfrac{A_1}{A_2}\right)^2 - 1\right]}}$$

The flow rate (R) i.e., the volume of the liquid flowing per second is given by $R = v_1 A_1$.

- ### Torricelli's Law of Efflux (Fluid Outflow)

As shown in the figure since the area of cross–section at A is very large as compared to that at orifice B, speed at A i.e. $v_A \approx 0$. Also the two fluid particles at A and B are at same pressure P_0 (atmospheric pressure). Applying Bernoulli's theorem at A and B.
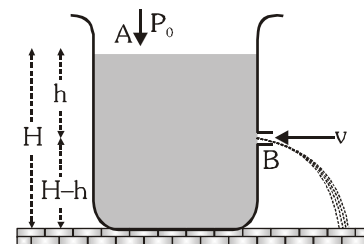
$$P_0 + \rho g H + \frac{1}{2}\rho v_A^2 = P_0 + \rho g(H-h) + \frac{1}{2}\rho v_B^2 \Rightarrow \frac{1}{2}\rho v_B^2 = \rho g h \Rightarrow v_B = \sqrt{2gh}$$

Equation is same as that of freely falling body after falling through h height and is known as Torricelli's law. Writing equation of uniformly accelerated motion in vertical direction

$$H - h = 0 + \frac{1}{2}gt^2 \text{ (from } s_y = u_y t + \frac{1}{2}a_y t^2) \Rightarrow t = \sqrt{\frac{2(H-h)}{g}}$$

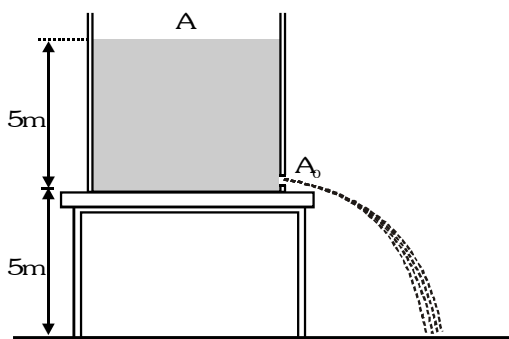Horizontal range $\quad R = v_x t = \sqrt{2gh}\ \sqrt{\dfrac{2(H-h)}{g}} = 2\sqrt{h(H-h)}$

Range R will be maximum when R^2 is maximum. i.e.,

$$\frac{d}{dh}\left(R^2\right) = 0 \Rightarrow 4\,\frac{d}{dh}(Hh-h^2) = 0 \Rightarrow H-2h = 0, \text{ i.e., } h = \frac{H}{2}$$

$$\therefore\ R_{maximum} = 2\sqrt{\frac{H}{2}\left[H - \frac{H}{2}\right]} = H$$

Example

A cylindrical tank 1 m in radius rests on a platform 5 m high. Initially the tank is filled with water upto a height of 5 m. A plug whose area is 10^{-4} m² is removed from an orifice on the side of the tank at the bottom. Calculate

(a) initial speed with which the water flows from the orifice

(b) initial speed with which the water strikes the ground

Solution

(a) Speed of efflux $v_H = \sqrt{(2gh)} = \sqrt{2 \times 10 \times 5} \simeq 10$ m/s

(b) As initial vertical velocity of water is zero, so its vertical velocity when it hits the ground

$$v_v = \sqrt{2gh} = \sqrt{2 \times 10 \times 5} \simeq 10 \text{ m/s}$$

So the initial speed with which water strikes the ground, $v = \sqrt{v_H^2 + v_V^2} = 10\sqrt{2} = 14.1$ m/s

Example

In a given arrangement

(a) Find out velocity of water coming out of 'C'

(b) Find out pressure at A, B and C.

Solution

(a) Applying Bernoulli's equation between liquid surface and point 'C'.

$$p_a + \frac{1}{2}\rho v_1^2 = p_a - \rho g h_3 + \frac{1}{2}\rho v_2^2$$

through continuity equation $\quad Av_1 = av_2 \;,\; v_1 = \dfrac{av_2}{A} \Rightarrow \dfrac{1}{2}\rho\dfrac{a^2}{A^2}v_2^2 = -\rho g h_3 + \dfrac{1}{2}\rho v_2^2$

$$v_2^2 = \frac{2gh_3}{1 - \dfrac{a^2}{A^2}} \;,\; v_2 = \sqrt{\frac{2gh_3}{1 - \dfrac{a^2}{A^2}}}$$

(b) Pressure at A just outside the tube $p_A = p_{atm} + \rho g h_1$
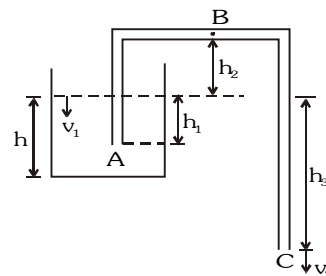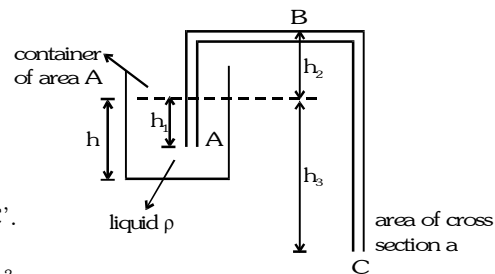
For pressure at B : $\qquad p_A + 0 + 0 = p_B + \rho g h_2 + \dfrac{1}{2}\rho v_B^2$

$$p_B = P_A - \rho g h_2 - \frac{1}{2}\rho\left(\frac{2gh_3}{1 - \dfrac{a^2}{A^2}}\right)$$

Pressure at C : $\qquad p_C = p_{atm}$

Example

A tank is filled with a liquid upto a height H. A small hole is made at the bottom of this tank. Let t_1 be the time taken to empty first half of the tank and t_2 time taken to empty rest half of the tank, then find $\dfrac{t_1}{t_2}$.

Solution

Let at some instant of time the level of liquid in the tank is y. Velocity of efflux at this instant of time $v = \sqrt{2gy}$.

Now, at this instant volume of liquid coming out the hole per second is $\left(\dfrac{dV_1}{dt}\right)$

Volume of liquid coming down in the tank per second is $\left(\dfrac{dV_2}{dt}\right)$

$$\frac{dV_1}{dt} = \frac{dV_2}{dt} \Rightarrow av = A\left(\frac{-dy}{dt}\right) \qquad \therefore\; a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right) \;\;...(i)$$

(Here area of cross-section of hole and tank are respectively a and A)

Substituting the proper limits in equation (i),

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}}\int_H^{H/2} y^{-1/2}dy \Rightarrow t_1 = \frac{2A}{a\sqrt{2g}}\left[\sqrt{y}\right]_{H/2}^{H} = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right] = \frac{A}{a}\sqrt{\frac{H}{g}}\left(\sqrt{2} - 1\right) \;\;...(ii)$$

Similarly, $\qquad \displaystyle\int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}}\int_{H/2}^{0} y^{-1/2}dy \Rightarrow t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \;\;...(iii)$

From equations (ii) and (iii), $\dfrac{t_1}{t_2} = \sqrt{2} - 1 = 0.414$

Example

A fixed container of height ' H ' with large cross-sectional area ' A ' is completely filled with water. Two small orifice of cross-sectional area ' a ' are made, one at the bottom and the other on the vertical side of the container at a distance H/2 from the top of the container. Find the time taken by the water level to reach a height of H/2 from the bottom of the container.

Solution

$v_1 = \sqrt{2g(h - H/2)}$; $v_2 = \sqrt{2gh}$ $\qquad \therefore$ By continuity equation

$A\left(-\dfrac{dh}{dt}\right) = a(v_1 + v_2) \Rightarrow A\left(-\dfrac{dh}{dt}\right) = a\left\{\sqrt{2g(h - H/2)} + \sqrt{2gh}\right\}$

or $-\dfrac{A}{a\sqrt{2g}} \displaystyle\int_{H}^{H/2} \dfrac{dh}{\sqrt{h} + \sqrt{h - H/2}} = \int_0^t dt \Rightarrow t = \dfrac{2A}{3a}\left(\sqrt{2} - 1\right)\sqrt{\dfrac{H}{g}}$
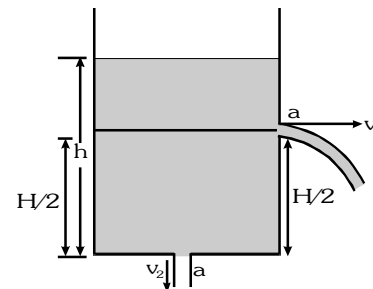
Example

A cylindrical vessel filled with water upto a height of 2 m stands on a horizontal plane. The side wall of the vessel has a plugged circular hole touching the bottom. Find the minimum diameter of the hole so that the vessel begins to move on the floor if the plug is removed. The coefficient of friction between the bottom of the vessel and the plane is 0.4 and total mass of water plus vessel is 100 kg.

Solution

From Torricelli's theorem, velocity of efflux = $\sqrt{2gh}$

Momentum per second carried by water stream
= density   volume coming out per second   velocity = ρ   av   v = ρav²
Hence force on cylindrical vessel = ρa2gh

Cylinder starts to move when reaction force is just equal to maximum force of friction.

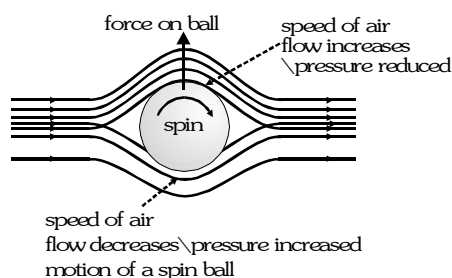i.e., μMg=ρa2gh $\Rightarrow$ a = $\dfrac{\mu M}{2\rho h}$ = $\dfrac{0.4 \times 100}{2 \times 10^3 \times 2}$ = 0.01 m²

Area of circular hole = $\dfrac{\pi d^2}{4}$ = 0.01 m² $\Rightarrow$ d = $\sqrt{\dfrac{0.01 \times 4}{\pi}}$ = 0.113m

EXAMPLES OF BERNOULLI'S THEOREM

1   **Magnus Effect (Spinning Ball)**

Tennis and cricket players usually experience that when a   ball is thrown spinning it moves along a curved path. This is called *swing* of the ball. This is due to the air which is being dragged round by the spinning ball. When the ball spins, the layer of the air around it also moves with the ball. So, as shown in figure the resultant velocity of air increases on the upper side and reduces on the lower side.

force on ball
speed of air
flow increases
\pressure reduced
spin
speed of air
flow decreases\pressure increased
motion of a spin ball

Hence according to Bernoulli's theorem the pressure on the upper side becomes lower than that on the lower side. This pressure difference exerts a force on the ball due to which it moves along a curved path. This effect is known as *Magnus–effect*.

1   Motion of the Ping–Pong Ball

When a ping–pong ball is placed on a vertical stream of water– fountain, it rises upto a certain height above the nozzle of the fountain and spins about its axis.

The reason for this is that the streams of water rise up from the fountain with very large velocity so that the air–pressure in them decreases. Therefore, whenever the ball goes out from the streams, the outer air which is at atmospheric pressure pushes it back into the streams (in the region of low pressure). Thus the ball remains in *stable equilibrium* on the fountain.
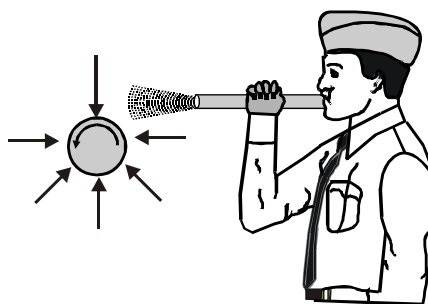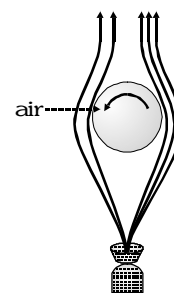
If we blow air at one end of a narrow tube, the air emerges from the other end at high speed and so the pressure falls there. If a ping–pong ball is left free slightly below this end, it does not fall down due to the large pressure (atmospheric) below the ball. Similarly, if we blow air in between two ping–pong balls suspended by light threads near each other, the balls come close to each other due to the decrease of air pressure between them. Same is the reason that when air is blown below a pan of a physical balance the pan is depressed down.

1   Aerofoil

This is a device which is shaped in such a way so that the relative motion between it and a fluid produces a force perpendicular to the flow. As shown in the figure the shape of the aerofoil section causes the fluid to flow faster over the top surface then over the bottom i.e. the streamlines are closer above than below the aerofoil. By Bernoullis theorem the pressure above is reduced and that underneath is increased.

high speed, reduced pressure

aerofoil

lift

low speed, increased pressure

principle of an aerofoil
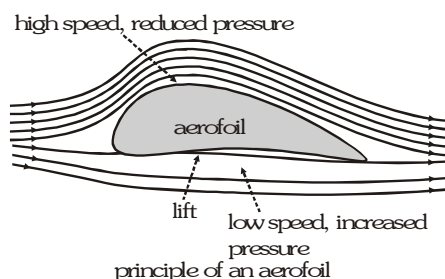
Thus a resultant upward force is created normal to the flow and it is this force which provides most of lift upward force for an aeroplane. Examples of aerofoils are aircraft wings, turbine blades and propellers.

1   Pull–in or Attraction Force by Fast Moving Trains

If we are standing on a platform and a train passes through the platform with very high speed we are pulled towards the train. This is because as the train comes at high speed, the pressure between us and the train decreases. Thus the air behind us which is still at atmospheric pressure pushes us towards the train. The reason behind flying–off of small papers, straws and other light objects towards the train is also the same.

◗ **Sprayer or Atomizer**

This is an instrument used to spray a liquid in the form of small   droplets (fine spray). It consists of a vertical tube whose lower end is dipped in the liquid to be sprayed, filled in a vessel.
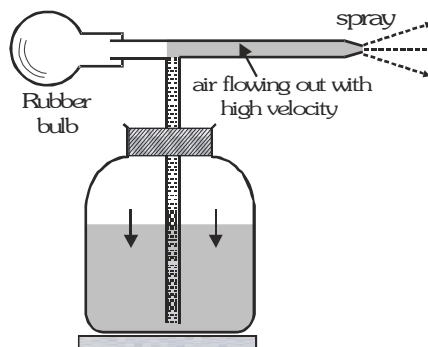

The upper end opens in a horizontal tube. At one end of the horizontal tube there is a rubber bulb and at the other end there is a fine bore (hole). When the rubber bulb is squeezed, air rushes out through the horizontal tube with very high velocity and thus the pressure reduces (according to Bernoulli's theorem). So the liquid rises and comes out through narrow end in form of droplets. It is used in spray gun, perfumes, deodorant and etc.

◗ **Bunsen's Burner (Jet)**

It is based on the working of jet. It consists of a long tube having a fine nozzle O at the bottom. The burning gas enters the tube through O and burns in a flame at the top of the tube.

To produce a non–luminous flame, the air of the atmosphere is mixed with the gas. Since the nozzle O is fine, the gas enters with a large velocity and so the pressure inside the tube is lowered than the outer atmospheric pressure. Therefore, air from outside rushes into the tube through a hole and is mixed with the burning gas.
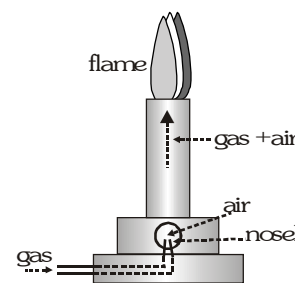

◗ **Filter Pump (Nozzle)**

It is based on the working of nozzle. It consists of a wide tube M N in the upper part of which there is another tube A. The upper end of A is connected to a water tank, while its lower end has a fine bore through which water comes out in the form of a jet. The vessel which is to be evacuated is connected to the tube M N as shown.

The velocity of the emerging water jet is very large. Therefore, the pressure of the air near the jet becomes less than the pressure in the vessel. Hence air from the vessel rushes into the tube M N and is carried out along with the water jet. Thus partial vacuum is created in the vessel.
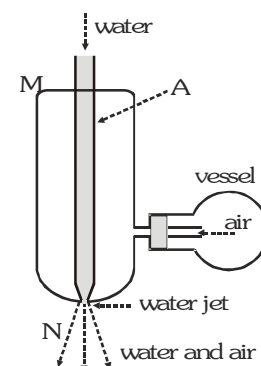
◗ **Blowing–off of Tin Roof Tops in Wind Storm**

When wind blows with a high velocity above a tin roof, it causes lowering of pressure above the roof, while the pressure below the roof is still atmospheric. Due to this pressure–difference the roof is lifted up.
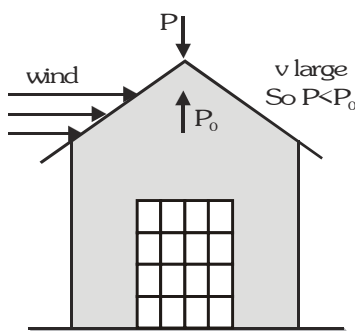
VISCOSITY

Viscosity is the property of the fluid (liquid or gas) by virtue of which it opposes the relative motion between its adjacent layers. It is the fluid friction or internal friction. The internal tangential force which try to retard the relative motion between the layers is called viscous force.

NEWTON'S LAW OF VISCOSITY

Suppose a liquid is flowing in streamlined motion on a horizontal surface OX. The liquid layer in contact with the surface is at rest while the velocity of other layers increases with increasing distance from the surface OX. The highest layer flows with maximum velocity.

horizontal surface
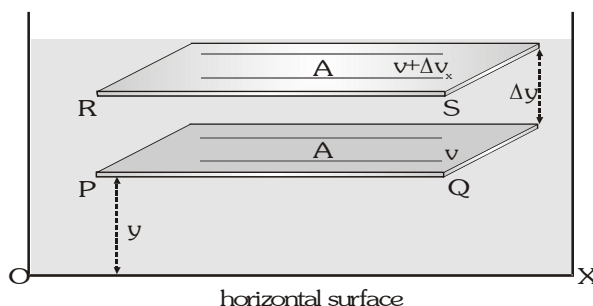
Let us consider two parallel layers PQ and RS at distances y and $y + \Delta y$ from OX. Thus the change in velocity in a perpendicular distance Δy is Δv_x. The rate of change of velocity with distance perpendicular to the direction

of flow i.e. $\dfrac{\Delta v_x}{\Delta y}$, is called velocity–gradient. According to Newton, the viscous force F acting between two layers

of a liquid flowing in streamlined motion depends upon following two factors :

(i)   F $\propto$ Contact–area of the layers   i.e. F $\propto$ A

(ii)   F $\propto$ Velocity–gradient $\left[\dfrac{\Delta v_x}{\Delta y}\right]$ between layers i.e. F $\propto$ $\dfrac{\Delta v_x}{\Delta y}$. Therefore F $\propto$ A $\dfrac{\Delta v_x}{\Delta y}$ $\Rightarrow$ F = ηA $\dfrac{\Delta v_x}{\Delta y}$ where η is

a constant called coefficient of viscosity of the liquid.

In above formula if A = 1 and $\dfrac{\Delta v_x}{\Delta y}$ = 1, then F = η. i.e. the coefficient of viscosity of a liquid is defined as the

viscous force per unit area of contact between two layers having a unit velocity gradient.

- • SI UNITS   : $\dfrac{N-s}{m^2}$ or deca poise

- • CGS UNITS   : dyne–s/cm^2 or poise (1 decapoise = 10 poise)

- • Dimension   : $M^1L^{-1}T^{-1}$

Example

A plate of area 2 m^2 is made to move horizontally with a speed of 2 m/s by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1m and the liquid in contact with the plate is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.

E

Solution

Velocity gradient $= \dfrac{\Delta v}{\Delta y} = \dfrac{2-0}{1-0} = 2 \dfrac{m/s}{m} = 2s^{-1}$

From, Newton's law of viscous force,

$|F| = \eta A \dfrac{\Delta v}{\Delta y} = (0.01 \times 10^{-1})(2)(2) = 4 \times 10^{-3}$ N
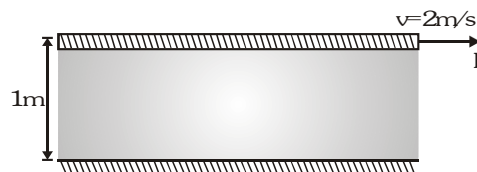
Example

A man is rowing a boat with a constant velocity v_0 in a river the contact area of boat is 'A' and coefficient of viscosity is η. The depth of river is 'D'. Find the force required to row the boat.
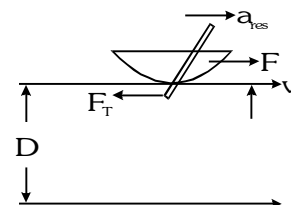
Solution

$F - F_T = m\,a$

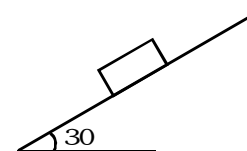As boat moves with constant velocity $a = 0$ so $F = F_T$

But $F_T = \eta A \dfrac{dv}{dz}$ but $\dfrac{dv}{dz} = \dfrac{v_0 - 0}{D} = \dfrac{v_0}{D}$

then $F = F_T = \dfrac{\eta A v_0}{D}$

Example

A cubical block (of side 2m) of mass 20 kg slides on inclined plane lubricated with the oil of viscosity $\eta = 10^{-1}$ poise with constant velocity of 10 m/s. Find out the thickness of layer of liquid. ($g = 10$ m/s²)

Solution
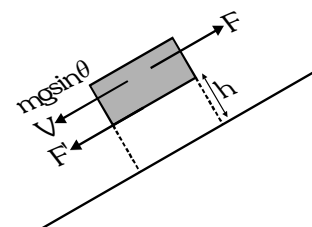
$F = F' = \eta A \dfrac{dv}{dz} = mg\,\sin\theta \qquad \dfrac{dv}{dz} = \dfrac{v}{h}$
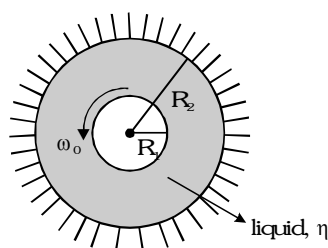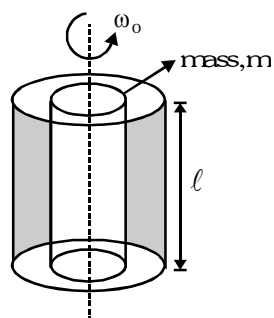
$\Rightarrow 20 \times 10 \times \sin 30 = \eta \times 4 \times \dfrac{10}{h}$

$\Rightarrow h = 4 \times 10^{-3}$ m = 4 mm

Example

As per the shown figure the central solid cylinder starts with initial angular velocity ω_0. Find out the time after which the angular velocity becomes half.
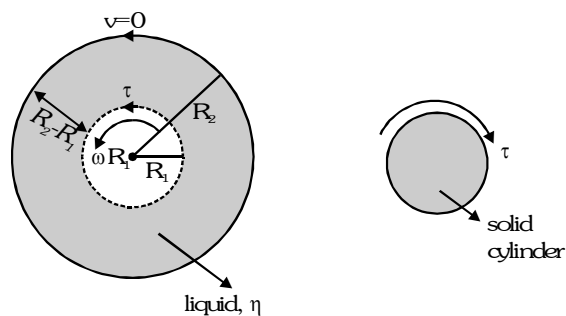
Solution

$$F = \eta \, A \, \frac{dv}{dz} \; , \; \text{where} \; \frac{dv}{dz} = \frac{\omega R_1 - 0}{R_2 - R_1}$$

$$F = \eta \, \frac{2\pi R_1 \, \ell \, \omega R_1}{R_2 - R_1} \; \text{and} \; \tau = FR_1 = \frac{2\pi \eta R_1^3 \, \omega \ell}{R_2 - R_1}$$

$$I \, \alpha = \frac{2\pi \eta R_1^3 \, \omega \ell}{R_2 - R_1} \Rightarrow \frac{MR_1^2}{2} \left(-\frac{d\omega}{dt} \right) = \frac{2\pi \eta R_1^3 \, \omega \ell}{R_2 - R_1}$$

$$\text{or} \quad -\int_{\omega_0}^{\omega_0/2} \frac{d\omega}{\omega} = \frac{4\pi \eta R_1 \ell}{m\,(R_2 - R_1)} \int_0^t dt \Rightarrow t = \frac{m\,(R_2 - R_1)\ell n 2}{4\pi \eta \ell R_1}$$

DEPENDENCY OF VISCOSITY OF FLUIDS

- ### On Temperature of Fluid

(a) Since cohesive forces decrease with increase in temperature as increase in K.E. Therefore with the rise in temperature, the viscosity of liquids decreases.

(b) The viscosity of gases is the result of diffusion of gas molecules from one moving layer to other moving layer. Now with increase in temperature, the rate of diffusion increases. So, the viscosity also increases. Thus, the viscosity of gases increases with the rise of temperature.

- ### On Pressure of Fluid

(a) The viscosity of liquids increases with the increase of pressure.

(b) The viscosity of gases is practically independent of pressure.

- ### On Nature of Fluid

STOKE'S LAW AND TERMINAL VELOCITY

Stoke showed that if a small sphere of radius r is moving with a velocity v through a homogeneous medium (liquid or gas), coefficient of viscosity η then the viscous force acting on the sphere is $F_v = 6\pi\eta r v$. It is called Stoke's Law.

- ### Terminal Velocity

When a solid sphere falls in a liquid then its accelerating velocity is controlled by the viscous force of liquid and hence it attains a constant velocity which is known as terminal velocity (v_T).

As shown in figure when the body moves with constant velocity i.e. terminal velocity (zero acceleration) the net upward force (upthrust Th + viscous force F_v) balances the downward force (weight of body W)

Therefore $\qquad$ Th + F_v = W $\Rightarrow \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_T = \frac{4}{3}\pi r^3 \rho g \Rightarrow v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)}{\eta}g$

where r = radius of body, ρ = density of body, σ = density of liquid,

η = coefficient of viscosity
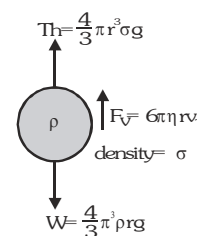
• **Some applications of Stoke's law :**

(a) The velocity of rain drops is very small in comparison to the velocity of a body falling freely from the height of clouds.

(b) Descending by a parachute with lesser velocity.

(c) Determination of electronic charge with the help of Milikan's experiment.

Example

A spherical ball is moving with terminal velocity inside a liquid. Determine the relationship of rate of heat loss with the radius of ball.

Solution

Rate of heat loss = power = $F \quad v = 6 \pi \eta \; rv \quad v = 6 \pi \eta rv^2 = 6p\eta r \left[\dfrac{2}{9} \dfrac{gr^2(\rho_0 - \rho_\ell)}{\eta} \right]^2$

Therefore rate of heat loss $\propto r^5$

Example

A drop of water of radius 0.0015 mm is falling in air. If the coefficient of viscosity of air is $1.8 \quad 10^{-5}$ kg m^{-1} s^{-1}. What will be the terminal velocity of the drop. Density of air can be neglected.

Solution

$$v_T = \frac{2}{9} \frac{r^2(\rho - \sigma)g}{\eta} = \frac{2 \times \left[\dfrac{15 \times 10^{-4}}{1000} \right]^2 \times 10^3 \times 9.8}{9 \times 1.8 \times 10^{-5}} = 2.72 \times 10^{-4} \text{ m/s}$$

REYNOLDS NUMBER (R_e)

The type of flow pattern (laminar or turbulent) is determined by a non-dimensional number called Reynolds number (R_e). Which is defined as $R_e = \dfrac{\rho v d}{\eta}$ where ρ is the density of the fluid having viscosity η and flowing with mean speed v. d denotes the diameter of obstacle or boundary of fluid flow.

Although there is not a perfect demarkation for value of R_e for laminar and turbulent flow but some authentic references take the value as

| R_e	< 1000	> 2000	between 1000 to 2000
Type of flow	laminar	often turbulent	may be laminar or turbulent

on gradually increasing the speed of flow at certain speed transition from laminar flow to turbulent flow takes place. This speed is called critical speed. For lower density and higher viscosity fluids laminar flow is more probable.

SOME WORKED OUT EXAMPLES

Example#1

The pressure of water in a water pipe when tap is opened and closed is respectively 3×10^5 Nm^{-2} and 3.5×10^5 Nm^{-2}. With open tap, the velocity of water flowing is
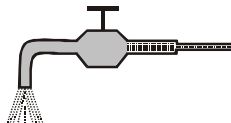
(A) 10 m/s   (B) 5 m/s   (C) 20 m/s   (D) 15 m/s
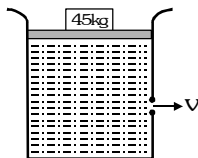
Solution   **Ans. (A)**

$$P_{open} + \frac{1}{2}\rho v^2 = P_{closed} \Rightarrow v = \sqrt{\frac{2\left(P_{closed} - P_{open}\right)}{\rho}} = \sqrt{\frac{2 \times (3.5-3) \times 10^5}{10^3}} = 10 \text{ m/s}$$

Example#2

A large cylindrical tank of cross-sectional area 1m^2 is filled with water. It has a small hole at a height of 1m from the bottom. A movable piston of mass 5 kg is fitted on the top of the tank such that it can slide in the tank freely. A load of 45 kg is applied on the top of water by piston, as shown in figure. The value of v when piston is 7m above the bottom is (g = 10 m/s^2)

(A) $\sqrt{120}$ m/s   (B) 10 m/s   (C) 1 m/s   (D) 11 m/s

Solution   **Ans. (D)**

$$\frac{1}{2}\rho v^2 = \rho g h + \frac{Mg}{A} \Rightarrow v = \sqrt{2gh + \frac{2Mg}{\rho A}} = \sqrt{2 \times 10 \times 6 + \frac{2 \times 50 \times 10}{10^3 \times 1}} = \sqrt{120+1} = \sqrt{121} = 11\text{m/s}$$

Example#3

An open vessel full of water is falling freely under gravity. There is a small hole in one face of the vessel, as shown in the figure. The water which comes out from the hole at the instant when hole is at height H above the ground, strikes the ground at a distance of x from P. Which of the following is correct for the situation described?

(A) The value of x is $2\sqrt{\dfrac{2hH}{3}}$

(B) The value of x is $\sqrt{\dfrac{4hH}{3}}$

(C) The value of x can't be computed from information provided.

(D) The question is irrelevant as no water comes out from the hole.
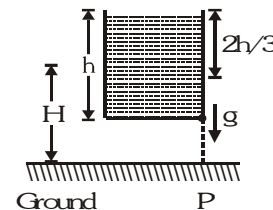
Solution   **Ans. (D)**

As vessel is falling freely under gravity, the pressure at all points within the liquid remains the same as the atmospheric pressure. If we apply Bernoulli's theorem just inside and outside the hole, then

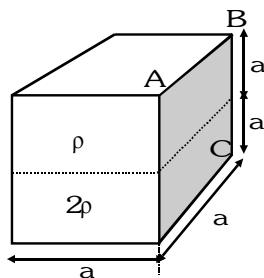$$P_{inside} + \frac{\rho v_{inside}^2}{2} + \rho g_{eff} y = p_{outside} + \frac{\rho v_{outside}^2}{2} + \rho g_{eff} y$$

$$v_{inside} = 0 \text{ , } p_{inside} = p_{outside} = p_0 \text{ [atmospheric pressure]}$$

Therefore, $v_{outside} = 0$. i.e., no water comes out.

E

Example#4

A cuboid (a × a × 2a) is filled with two immiscible liquids of density 2ρ & ρ as shown in the figure. Neglecting atmospheric pressure, ratio of force on base & side wall of the cuboid is

(A) 2:3   (B) 1:3   (C) 5:6   (D) 6:5

Solution   **Ans. (D)**

$$F_b = \left(2\rho gh + \rho gh\right)a^2 = 3\rho gha^2 \; ; F_w = \left[\frac{\rho gh}{2}\right]ah + \left(2\rho gh\right)ah = \frac{5}{2}\rho gha^2 \text{ [here } h=a]; \; \frac{F_b}{F_w} = \frac{3}{5/2} = \left(\frac{6}{5}\right)$$

Example#5

During blood transfusion the needle is inserted in a vein where the gauge pressure is 2000 Pa. At what height must the blood container be placed so that blood may just enter the vein ? [Density of whole blood = 1.06 × 10^3 kg m^{-3}].

(A) 0.192 m   (B) 0.182 m   (C) 0.172 m   (D) 0.162 m

Solution   **Ans. (A)**

Pressure $P = h\rho g \;\; \Rightarrow \;\; h = \dfrac{2000}{1.06 \times 10^3 \times 9.8} = 0.192 m$

Example#6

A liquid of density ρ is filled in a U-tube, whose one end is open & at the other end a bulb is fitted whose pressure is P_A. Now this tube is moved horizontally with acceleration 'a' as shown in the figure. During motion it is found that liquid in both column is at same level at equilibrium. If atmospheric pressure is P_0, then value of P_A is
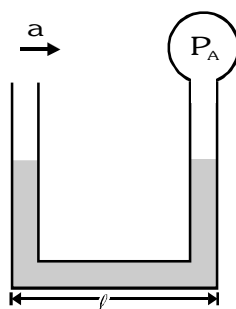
(A) ρaℓ   (B) ρgℓ   (C) P_0–ρaℓ   (D) P_0 + ρaℓ
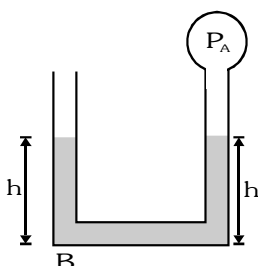
Solution   **Ans. (C)**
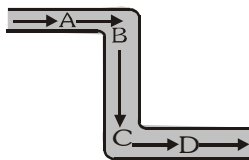
Consider a point B at the horizontal portion of tube, pressure from both side should be same.

∴ P_0 + ρgh = P_A + ρgh + ρaℓ, P_A = P_0 – ρaℓ

Example#7

The tube shown is of non-uniform cross-section. The cross-section area at A is half of the cross-section area at B,C and D. A liquid is flowing through in steady state.The liquid exerts on the tube-

Statement I : A net force towards right.   **Statement II :** A net force towards left.
Statement III : A net force in some oblique direction.   **Statement IV :** Zero net force
Statement V: A net clockwise torque.   **Statement VI:** A net counter-clockwise torque.
Out of these
(A) Only statement I and V are correct   (B) Only statement II and VI are correct
(C) Only statement IV and VI are correct   (D) Only statement III and VI are correct

Solution   **Ans. (A)**

The force has been exerted by liquid on the tube due to change in momentum at the corners i.e., when liquid is taking turn from A to B and from B to C.As corss-section area at A is half of that of B and C, so velocity of liquid flow at B and C is half to that of velocity at A. Let velocity of flow of liquid at A be v and cross section area

at A be S, the velocity of flow of liquid at B and C would be $\frac{v}{2}$ [from continuity equation] and corss section area

at B and C would be 2S.
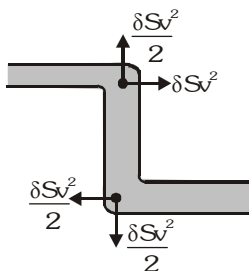
Due to flow of liquid, it is exerting a force per unit time of ρSv^2 on the tube, where ρ is the density of liquid, S is cross section areaand v is velocity of flow of liquid. The force exerted by liquid on tube is shown in the figure. Which clearly shows that a net force is acting on tube due to flowing liquid towards right and a clockwise torque sets in

Example#8

A rigid ring A and a rigid thin disk B both made of same material, when gently placed on water, just manage to float due to surface tension as shown in the figure. Both the ring and the disk have same radius. What can you conclude about their masses?
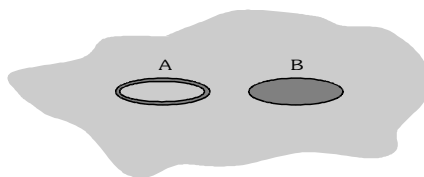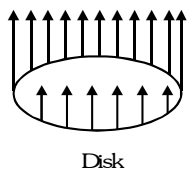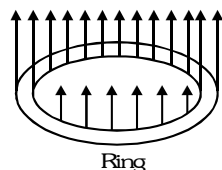

(A) Both have the same mass.   (B) Mass of the ring is half of that of the disk.
(C) Mass of the ring is double to that of the disk.   (D) More information is needed to decide.

Solution   **Ans. (C)**
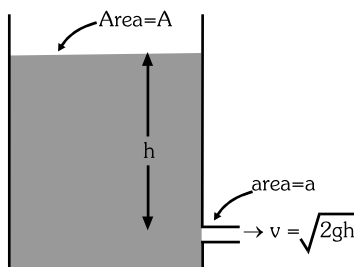

Ring have double surface than that of disk.

E

Example#9

A hole is made at the bottom of a large vessel open at the top. If water is filled to a height h, it drains out completely in time t. The time taken by the water column of height 2h to drain completely is

(A) $\sqrt{2}t$ (B) 2t   (C) $2\sqrt{2}t$ (D) 4t

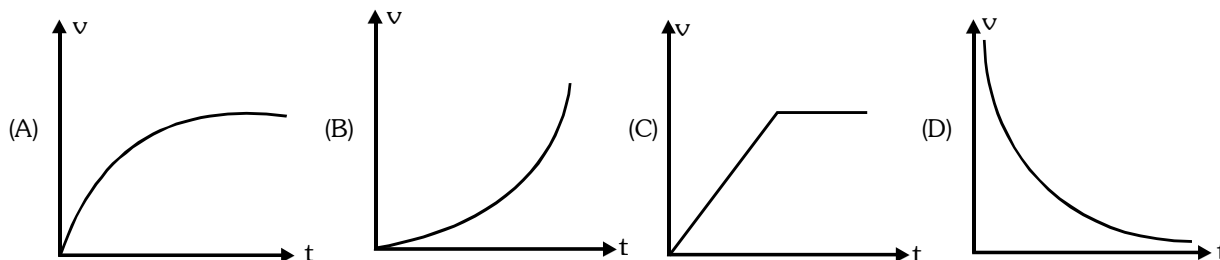**Solution**   **Ans. (A)**

Here $\left(-A\dfrac{dh}{dt}\right) = (av) = a\sqrt{2gh} \Rightarrow \int_0^h h^{-1/2}dh = \dfrac{a}{A}\sqrt{2g}\int_0^t dt \Rightarrow t = \dfrac{A}{a}\sqrt{\dfrac{2h}{g}} \Rightarrow t \propto \sqrt{h}$

Therefore for height 2h $t' \propto \sqrt{2h} \Rightarrow t' = \sqrt{2}\,t$

Example#10

A piece of cork starts from rest at the bottom of a lake and floats up. Its velocity v is plotted against time t. Which of the following best represents the resulting curve?

Solution   **Ans. (A)**

As the cork moves up, the force due to buoyancy remains constant. As its speed increases, the retarding force due to viscosity increases, being proportional to the speed. Thus the acceleration gradually decreases.

Example#11

A flat plate moves normally with a speed v_1 towards a horizontal jet of water of uniform area of cross-section. The jet discharges water at the rate of volume V per second at a speed of v_2. The density of water is ρ. Assume that water splashes along the surface of the plate at right angles to the original motion. The magnitude of the force acting on the plate due to jet of water is :–

(A) $\rho V v_1$ (B) $\rho\left(\dfrac{V}{v_2}\right)(v_1+v_2)^2$ (C) $\dfrac{\rho V}{v_1+v_2}v_1^2$ (D) $\rho V(v_1+v_2)$

Solution   **Ans. (D)**

$F = \dfrac{\Delta p}{\Delta t} = \dfrac{m(v_1+v_2)}{\Delta t} = V\rho(v_1+v_2)$ where $\dfrac{m}{\Delta t} = V = \text{volume/sec}$

Example#12

A tank full of water has a small hole at its bottom. Let t_1 be the time taken to empty first one third of the tank and t_2 be the time taken to empty second one third of the tank and t_3 be the time taken to empty rest of the tank then

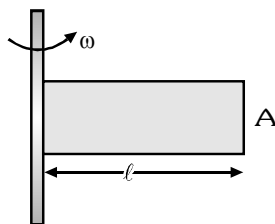(A) $t_1 = t_2 = t_3$ (B) $t_1 > t_2 > t_3$ (C) $t_1 < t_2 < t_3$ (D) $t_1 > t_2 < t_3$

Solution   **Ans. (C)**

As the height decreases, the rate of flow with which the water is coming out decreases.
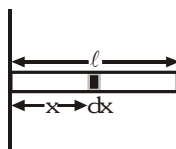
Example#13

A closed cylinder of length 'ℓ' containing a liquid of variable density $\rho(x) = \rho_0(1+\alpha x)$. Find the net force exerted by the liquid on the axis of rotation. (Take the cylinder to be massless and A = cross sectional area of cylinder)

(A) $\rho_0 A\omega^2\ell^2\left[\dfrac{1}{2} + \dfrac{1}{3}\alpha\ell\right]$
(B) $\rho_0 A\omega^2\ell^2\left[\dfrac{1}{2} + \dfrac{2}{3}\alpha\ell\right]$
(C) $\rho_0 A\omega^2\ell^2\left[\dfrac{1}{2} + \alpha\ell\right]$
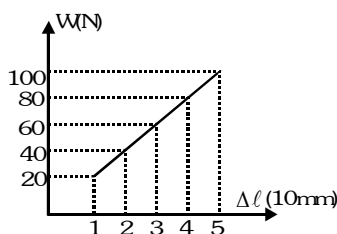(D) $\rho_0 A\omega^2\ell^2\left[\dfrac{1}{2} + \dfrac{4}{3}\alpha\ell\right]$

Solution **Ans. (A)**

$dm = \rho A dx$; $dF = (dm)\omega^2 x \Rightarrow F = \int_0^\ell \omega^2 x \rho A dx = \omega^2 \rho_0 A \int_0^\ell (1+\alpha x)x\, dx = \rho_0 A\omega^2\ell^2\left(\dfrac{1}{2} + \dfrac{1}{3}\alpha\ell\right)$

Example#14

The graph shows the extension of a wire of length 1m suspended from the top of a roof at one end and with a load W connected to the other end. If the cross sectional area of the wire is 1 mm², then the Young's modulus of the material of the wire is

(A) 2×10^{11} Nm^{-2}
(B) 2×10^{10} Nm^{-2}
(C) $\dfrac{1}{2} \times 10^{11}$ Nm^{-2}
(D) None of these

Solution **Ans. (B)**

$Y = \dfrac{F/A}{\Delta\ell/\ell} = \dfrac{W\ell}{A\Delta\ell} \Rightarrow \dfrac{W}{\Delta\ell} = \dfrac{YA}{\ell} = \text{slope} \Rightarrow Y = \dfrac{\ell}{A}(\text{slope}) = \dfrac{1}{10^{-6}}\left(\dfrac{40-20}{(2-1)\times 10^{-3}}\right) = 2\times 10^{10}\,\text{Nm}^{-2}$

Example#15

The diagram shows a force-extension graph for a rubber band.
Consider the following statements

I. It will be easier to compress this rubber than expand it.

II. Rubber does not return to its original length after it is stretched.

III. The rubber band will get heated if it is stretched and released.

Which of these can deduced from the graph-

(A) III only
(B) II and III
(C) I and III
(D) I only
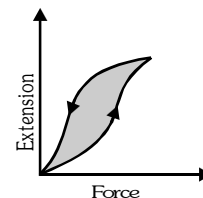
Solution **Ans. (A)**

Area of hysterisis loop gives the energy loss in the process of stretching and unstretching of rubber band and this loss will appear in the form of heating.

E

Example#16

A solid right cylinder of length ℓ stands upright at rest on the bottom of a large tub filled with water up to height h as shown in the figure-I. Density of material of the cylinder equals to that of water. Now the cylinder is pulled slowly out of water with the help of a thin light inextensible thread as shown in figure-II. Find the work done by the tension force develop in the thread.

Figure-I   Figure-II
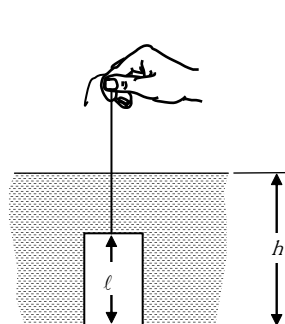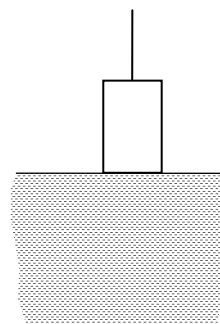
(A) mgh   (B) mgℓ (C) 0.5 mgℓ (D) mg(0.5ℓ+h)

Solution   **Ans. (C)**

Work done by tension force = work done by gravity = 0.5 mgℓ

Example#17

n drops of a liquid, each with surface energy E, joining to form a single drop

(A) some energy will be released in the process

(B) some energy will be absorbed in the process

(C) the energy released or absorbed will be $E(n-n^{2/3})$

(D) the energy released or absorbed will be $nE(2^{2/3}-1)$

Solution   **Ans. (A,C)**

$E = T\ 4\pi r^2 \Rightarrow \dfrac{4}{3}\pi R^3 = n \times \dfrac{4\pi}{3} r^3 \Rightarrow n = \dfrac{R^3}{r^3} \Rightarrow R = n^{1/3}r$

Surface energy of big drop $E' = T4\pi R^2 = T4\pi n^{2/3}r^2 = En^{2/3}$

Energy released $= nE - E' = nE - n^{2/3}E = E(n - n^{2/3})$

Example#18

A sphere is dropped into a viscous liquid of viscosity η from some height. If the density of material and liquid are ρ and σ respectively ($\rho > \sigma$) then which of the following is/are incorrect.

(A) The acceleration of the sphere just after entering the liquid is $g\left(\dfrac{\rho - \sigma}{\rho}\right)$

(B) Time taken to attain terminal speed $t \propto \rho^0$

(C) At terminal speed, the viscous force is maximum

(D) At terminal speed, the net force acting on the sphere is zero

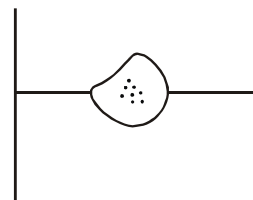**Solution**   **Ans. (A,B,C)**

Acceleration will be less than $g\left(\dfrac{\rho - \sigma}{\rho}\right)$. Time will depend on density ρ. Viscous force may be maximum or minimum both are possible

Example#19

Some pieces of impurity (density $=\rho$) is embedded in ice. This ice is floating in water (density $= \rho_w$). When ice melts, level of water will

(A) fall if $\rho > \rho_w$

(B) remain unchanged, if $\rho < \rho_w$

(C) fall if $\rho < \rho_w$

(D) rise if $\rho > \rho_w$

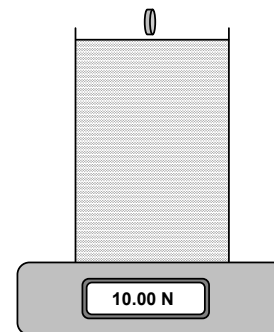**Solution**   **Ans. (A,B)**

Level will fall if initially the impurity pieces were floating along will ice and later it sinks.

Level will remain unchaged if initially they were floating and later also they keep floating.

Example#20

A glass full of water is placed on a weighing scale, which reads 10 N. A coin with weight 1 N is gently released into the water. At first, the coin accelerates as it falls and about halfway down the glass, the coin reaches terminal velocity. Eventually, the coin rests on the bottom of the glass. Acceleration due to gravity is 10 m/s². The scale reads

(A) 10.5 N when the coin accelerates at 5 m/s².

(B) 11.5 N, when the coin decelerates at 5 m/s².

(C) 11 N, when the coin moves with terminal velocity.

(D) 11 N, when the coin rests on the bottom.

Solution   **Ans. (ACD)**

When coin moves with terminal velocity or rests on the bottom   Reading = 10 + 1 = 11 N

When coin moves with acceleration of 5 m/s²   Reading = 10 + 1 (½) = 10.5 N

Example#21 to 23

Velocity of efflux in Torricelli's theorem is given by $v = \sqrt{2gh}$, here h is the height of hole from the top surface, after that, motion of liquid can be treated as projectile motion.
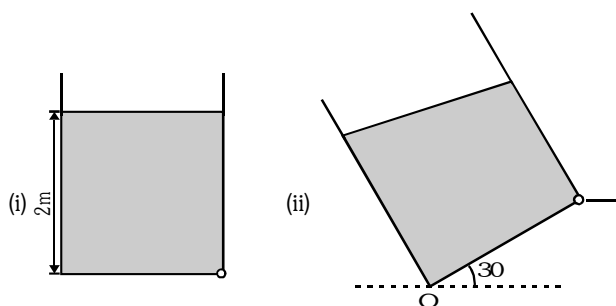
21. Liquid is filled in a vessel of square base (2m   2m) upto a height of 2m as shown in figure (i). In figure (ii) the vessel is tilted from horizontal at 30 . What is the velocity of efflux in this case. Liquid does not spills out?

(A) 3.29 m/s   (B) 4.96 m/s   (C) 5.67 cm   (D) 2.68 m/s

22. What is its time of fall of liquid on the ground?

(A) $\dfrac{1}{\sqrt{2}}$ s   (B) $\dfrac{1}{\sqrt{3}}$ s   (C) $\dfrac{1}{\sqrt{5}}$ s   (D) $\sqrt{2}$ s

23. At what distance from point O, will be liquid strike on the ground?

(A) 5.24 m   (B) 6.27 m   (C) 4.93 m   (D) 3.95 m

Solution

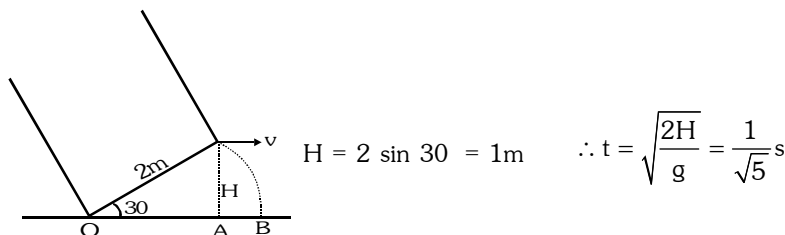**21.** **Ans. (B)**

The volume of liquid should remain unchanged. Hence, $2 \times 2 \times 2 = \frac{1}{2}\left[x + x + \frac{2}{\sqrt{3}} \right] \times 2$ $\therefore$ $x \approx 1.42 \, m$

Now $h = x \sin 60° = 1.23 \, m$ $\therefore v = \sqrt{2gh} = \sqrt{2 \times 10 \times 1.23} = 4.96 \, m/s$

22. **Ans. (C)**

$H = 2 \sin 30° = 1m$ $\therefore t = \sqrt{\dfrac{2H}{g}} = \dfrac{1}{\sqrt{5}} s$

23. **Ans. (D)**

$OA = 2\cos 30° - \sqrt{3}\,m \Rightarrow AB = vt = \dfrac{(4.96)}{\sqrt{5}} s$ $\therefore OB = 3.95 \, m$

Example#24 to 26

A cylindrical container of length L is full to the brim with a liquid which has mass density ρ. It is placed on a weight-scale; the scale reading is w. A light ball which would float on the liquid if allowed to do so, of volume V and mass m is pushed gently down and held beneath the surface of the liquid with a rigid rod of negligible volume as shown on the left.
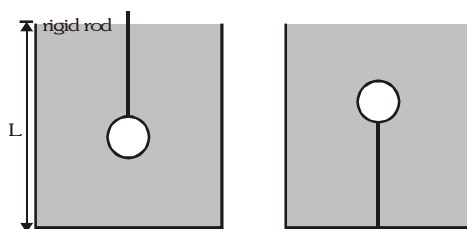
24. What is the mass M of liquid which overflowed while the ball was being pushed into the liquid?
(A) ρV (B) m   (C) $m - \rho V$ (D) none of these

25. What is the reading of the scale when the ball is fully immersed
(A) $w - \rho Vg$ (B) w   (C) $w + mg - \rho Vg$ (D) none of these

26. If instead of being pushed down by a rod, the ball is held in place by a thin string attached to the bottom of the container as shown on the right. What is the tension T in the string?
(A) $(\rho V - m)g$ (B) ρVg (C) mg   (D) none of these

24. **Ans. (A)**
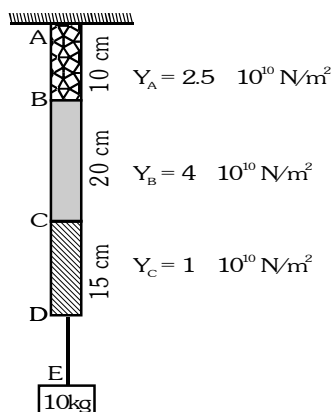Ball will displace V volume of liquid, whose mass in ρV.

25. **Ans. (C)**
Weight mg is entered while weight ρVg of liquid is overflowed.

26. **Ans. (A)**

$F = T + mg \Rightarrow V\rho g = T + mg \therefore T = (V\rho - m)g$
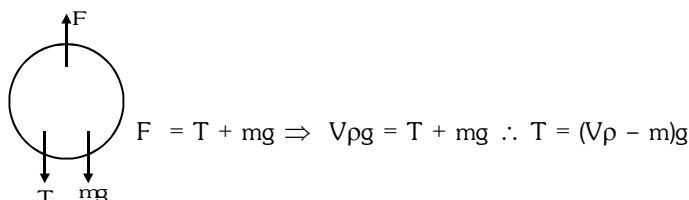
Example#28

A light rod of uniform cross-section of 10^{-4} m^2 is shown in figure. The rod consists of three different materials. Assume that the string connecting the block and rod does not elongate.

	Column I (Points)		Column II (Displacements in μm)	
(A)	B	(P)	4
(B)	C	(Q)	9
(C)	D	(R)	12
(D)	E	(S)	15
		(T)	24

Solution   **Ans.(A) P (B) Q (C) T (D) T**

$$\because Y = \frac{F/A}{\Delta \ell / \ell} \qquad \therefore \Delta \ell = \frac{F\ell}{YA} \qquad \Delta \ell_B = \frac{(100)(0.1)}{2.5 \times 10^{10} \times 10^{-4}} = 4 \times 10^{-6} m = 4 \mu m$$
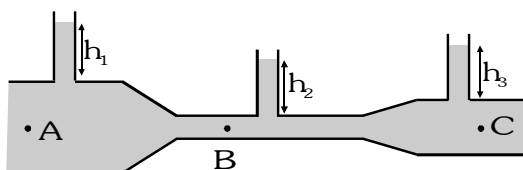
$$\Delta \ell_c - \Delta \ell_B = \frac{(100)(0.2)}{(4 \times 10^{10})(10^{-4})} = 5 \times 10^{-6} m = 5 \mu m \Rightarrow \Delta \ell_c = 9 \mu m$$

$$\Delta \ell_D - \Delta \ell_C = \frac{(100)(0.15)}{(1 \times 10^{10})(10^{-4})} = 15 \times 10^{-6} m = 15 \mu m \Rightarrow \Delta \ell_D = 24 \mu m$$

Displacement of E = Displacement of D = 24 μm

Example#29

An ideal liquid is flowing in a tube as shown in figure. Area of cross-section at points A, B and C are A_1, A_2 and A_3 respectively ($A_1 > A_3 > A_2$). v_1, v_2, > v_3 are the velocities at the points A, B and C respectively.

	Column I		Column II
(A)	h_1 is	(P)	Less than h_3
(B)	h_2 is	(Q)	More than h_3
(C)	v_1 is	(R)	Less than v_3
(D)	v_2 is	(S)	More than v_3
		(T)	The maximum value amongst the three velocities.

E

Solution   Ans. (A) Q (B) P (C) R (D) ST

By using $A_1v_1 = A_2v_2$ $\because$ $A_2 < A_3 < A_1$ $\therefore$ $v_2 > v_3 > v_1$. Now by using $P + \frac{1}{2}\rho v^2 + \rho gh$ = constant

We have $P_2 < P_3 < P_1$ so $h_1 > h_3$ and $h_2 < h_3$

Example#30

A square plate of 1m side moves parallel to a second plate with velocity 4 m/s. A thin layer of water exist between plates. If the viscous force is 2 N and the coefficient of viscosity is 0.01 poise then find the distance between the plates in mm.
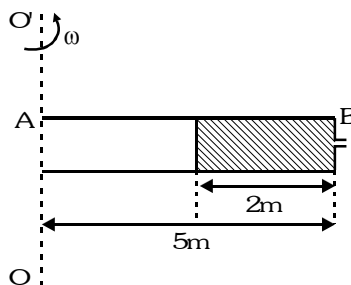
Solution   Ans. 2

$$F = \frac{\eta A v}{d} \Rightarrow d = \frac{\eta A v}{F} = \frac{0.01 \times 10^{-1} \times 1^2 \times 4}{0.002} = 2m$$

Example#31

A horizontal oriented tube AB of length 5 m rotates with a constant angular velocity 0.5 rad/s about a stationary vertical axis OO' passing through the end A. The tube is filled with ideal fluid. The end of the tube is open, the closed end B has a very small orifice. The velocity with which the liquid comes out from the hole (in m/s) is

Solution   Ans. 2

Apply Bernoullie theorem between two ends

$$P_0 + \rho\int_3^5 x\omega^2 dx = P_0 + \frac{1}{2}\rho v^2 \Rightarrow \frac{\rho\omega^2}{2}\left[5^2 - 3^2\right] = \frac{1}{2}\rho v^2 \Rightarrow (0.5)2 \quad 16 = v2 \Rightarrow v = 2m/s$$

Example#32

A ball of density ρ_0 falls from rest from a point P onto the surface of a liquid of density ρ in time T. It enters the liquid, stops, moves up, and returns to P in a total time 3T. Neglect viscosity, surface tension and splashing. Find
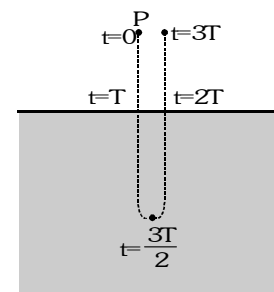
the ratio of $\dfrac{\rho}{\rho_0}$.

Solution   Ans. 3

It strike the surface of liquid with velocity $v_1 = gT$

In water (liquid) its time of flight $T = \dfrac{2v_1}{\left(\dfrac{\rho}{\rho_0}-1\right)g} = \dfrac{2gT}{\left(\dfrac{\rho}{\rho_0}-1\right)g} \Rightarrow \dfrac{\rho}{\rho_0} = 3$

Example#33

A cylinder fitted with piston as shown in figure. The cylinder is filled with water and is taken to a place where there is no gravity. Mass of the piston is 50 kg. The system is accelerated with acceleration 0.5 m/sec² in positive x–direction. Find the force exerted by fluid on the surface AB of the cylinder in decanewton. Take area of cross–section of cylinder to be 0.01 m² and neglect atmospheric pressure (1 decanewton =10N)
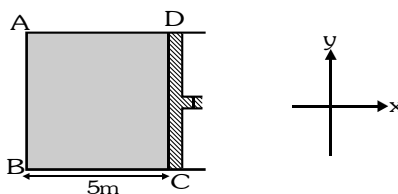
Solution   **Ans. 5**

Force due to piston = 50 × 0.5 = 25 N

Force due to fluid = (ρah) A=A (1000 × 5 × 0.5) = 2500 Pa × 0.01 m² = 25 N

Force on the surface AB = 50N = 5 decanewton

Example#34

A container filled with air under pressure P_0 contains a soap bubble of radius R. The air pressure has been reduced to half isothermally and the new radius of the bubble becomes $\dfrac{5R}{4}$. If the surface tension of the soap water solution is S, P_0 is found to be $\dfrac{12nS}{R}$. Find the value of n.

Solution   **Ans. 8**

$$\left(P_0 + \frac{4S}{R}\right)\left(\frac{4}{3}\pi R^3\right) = \left(\frac{P_0}{2} + \frac{4S \times 4}{5R}\right) \times \frac{4}{3}\pi\left(\frac{5R}{4}\right)^3 \Rightarrow P_0 + \frac{4S}{R} = \left(\frac{P_0}{2} + \frac{16S}{5R}\right)\frac{125}{64} \Rightarrow P_0 = \frac{96S}{R}$$

Example #35

Water (density ρ) is flowing through the uniform tube of cross-sectional area A with a constant speed v as shown in the figure. Find the magnitude of force exerted by the water on the curved corner of the tube is (neglect viscous forces)
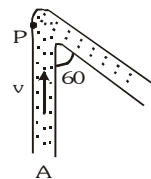
Solution

$$|\overrightarrow{\Delta P}_x| = mv\sin 60° = \frac{\sqrt{3}}{2}mv$$

$$|\overrightarrow{\Delta P}_y| = \frac{mv}{2} + mv = \frac{3}{2}mv \quad \Rightarrow \quad |\overrightarrow{\Delta P}_{net}| = \sqrt{\Delta P_x^2 + \Delta P_y^2} = \sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)}mv$$

$$|\overrightarrow{\Delta P}_{net}| = \sqrt{3}\,mv \Rightarrow |\overrightarrow{\Delta F}_{net}| = \sqrt{3}\left(\frac{dm}{dt}\right).v = \sqrt{3}\rho A v^2 \quad \left(\text{Since, } dm = A(v\,dt)\rho \Rightarrow \frac{dm}{dt} = A\rho v\right)$$